THE DYNAMIC FACTOR MODEL: 
AN APPLICATION TO INTERNATIONAL 
STOCK MARKET INTEGRATION\textsuperscript{1}

Bruno de Paula Rocha\textsuperscript{2} 
Rodrigo Marino Sekkel\textsuperscript{3}

Abstract

The Dynamic Common Factor Model has been largely used in recent macroeconometric studies. The model represents any vector of variables of interest through the sum of two non-observable orthogonal components – the common and the idiosyncratic one. One reason for the increasing interest in this model follows from its flexibility to work with large dimensional data set. This work makes an application of the model to a group of stock market indexes. Beyond presenting the model, we measure the dynamic integration of 24 different stock markets from Dec/1992 to May/2004.

\textbf{Key words:} Dynamic Factor Model and Stock Market.

\textbf{JEL Codes:} C32, C53 and G15.

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\textsuperscript{2} PhD student at IPE-USP, brunor@usp.br.

\textsuperscript{3} Msc in Economics at IPE-USP, rsekkel@usp.br.
1. Introduction

Since the end of the 80’s and beginning of the 90’s, the world economy has experienced an ever-increasing interdependence of its stock markets. After the huge technology progress and expressive growth of capital flows of the last decades, worldwide stock markets have become much more sensitive to local events, as the financial crisis from the late 90’s have painfully proven.

After the crash of 1987 of the New York Stock Exchange, innumerable articles have analyzed the pattern of co-movement between stock exchanges. The first stream of the literature tried to analyze the dynamics of that co-movement by checking the correlation between the various markets, as Füerstenberg and Jeon (1989), Bertero and Mayer (1990), Hamao et. all. (1990), Koch and Koch (1991), and Cheung and Ng (1992).

Another stream of the literature focused on searching for common stochastic trends and checking the transmission of shocks between markets, very often finding conflicting results. Kasa (1992) is one of the first studies to apply Johansen (1988) methodology to test for a common stochastic trend between stock markets. Using monthly and quarterly observations from several stock exchanges, as USA, United Kingdom, Japan, Canada and Germany for the period of 1974 to 1990, the author finds one common stochastic trend between those markets. Blackman et. all. (1994), Mashi and Mashi (1997), Jochum et. all. (1999), among many others4, also find evidence of a common stochastic trend between stock exchanges of developed and developing countries. Richards (1995), on the other hand, criticize that literature for ignoring the small sample properties of the estimators. Applying the small sample critical values of Cheung and Lai (1993), or after generating the critical values by Monte Carlo simulation, the author is not able to reject the null hypothesis of no cointegration.

Many authors have argued that the increasing integration of financial markets could jeopardize investors search for risk diversification. As pointed by Garrett and Spyrou (1999), that may not be the case. Small

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or statistically insignificant long-run coefficients imply the unimportance of some countries to the maintenance of the common-trend, thus making long-run diversification between countries possible. The measure of integration here proposed permits us to measure the integration at all phases of the spectrum, consequently allowing us to detach the relative importance of short and long-run co-movements.

Sala (2001) states that the cointegration analysis, seen under the frequency domain, is equivalent to the study of the spectral density at the zero frequency. The methodology proposed in this study allows us to benefit from the information of higher frequencies of the spectrum, also capturing short-term coincident movements.

In this sense, it is the objective of this paper to propose a methodology to infer the pattern of dynamic integration for a set of 24 stock market exchanges of 21 different nations. For that purpose, we apply Forni et al (2000) dynamic factor model. The model represents any vector of series through the sum of two orthogonal non-observable components – the common and the idiosyncratic one. The idea underlying the decomposition is that the behavior of a relatively large set of variables is driven by a possibly small number of common factors and idiosyncratic shocks. In a spirit close to Forni and Reichlin (1999), the percentage of variance explained by the common component is our measure of dynamic integration.

Besides this short introduction, this work consist of three more sections. In the next one, the methodology and the dynamic factor model are described. Next, in the third section, the main results are listed and discussed. Finally, section four summarizes and final considerations are presented.

2. The Dynamic Factor Model

The procedure chosen to measure the international stock markets integration is an application of the model first developed by Forni and Reichlin (1999). According to the authors, any vector may be decomposed into the sum of two unobservable and orthogonal components:

\[ x_i^t = \chi_i^t + \varepsilon_i^t \]  (1)
Where the sequence \( \{ x_t^i ; t \in \mathbb{N} \text{ and } i = 1,...,n \} \) represents the series under study, \( \{ x_t^i ; t \in \mathbb{N} \text{ and } i = 1,...,n \} \) represents the common component and \( \{ \epsilon_t^i ; t \in \mathbb{N} \text{ and } i = 1,...,n \} \) represents the idiosyncratic shocks.

As the components \( x_t^i \) and \( \epsilon_t^i \) are orthogonal, we take the fraction of the total variance explained by the common component as a measure of dynamic integration of \( x_t^i \). The strategy for the estimation of the latent components is based on an application of the Common Dynamic Factor Model [Forni, Hallin, Lippi and Reichlin (2000)], which we briefly describe below.

First, we define the vector of common component for the linear combination:

\[
\chi_t^i = A_t^i(L)u_t
\]  

(2)

Where \( \{ u_t = (u_{1t}, u_{2t},...,u_{qt}) ; t \in \mathbb{N} \text{ and } q<<n \} \) is the common shocks vector and \( A_t^i(L) \) is a rational matrix function in the lag operator \( L \). The idea behind the model is that “the behavior of several variables is driven by few common forces, the factors, plus idiosyncratic shocks” [Favero, Marcellino and Neglia (2002): 3]. The vector of shocks, or common factors, \( u_t \), defines the common dynamics. Although it is common to all variables, it has different effects on each of them, according to the loading coefficients of the \( A_t^i(L) \) matrix.

The model presented by Forni, Hallin, Lippi and Reichlin (2000) is a generalization of a large class of models introduced in the macroeconomic literature by Sargent and Sims (1977) and Geweke (1977), as well as Chamberlain and Rotstock (1983). The framework proposed in the 70’s also allows dynamic, though it requires finite cross-section dimension and orthogonal idiosyncratic components. The static model proposed by Chamberlain and Rotstock (1983) requires orthogonality of the idiosyncratic components, but it allows infinite cross-section dimension. The dynamic model proposed by Forni, Hallin, Lippi and Reichlin (2000) is more general, as it allows infinite cross-section dimension and non-orthogonal idiosyncratic components.

In order to estimate the common components, Forni, Hallin, Lippi and Reichlin (2000) establish a set of assumptions about the variables. Next we describe the first set of assumptions.
Assumption 1:

1.1 The stochastic process \( \{u_t = (u_{1t}, u_{2t}, \ldots, u_{qt}); t \in \mathbb{N} \text{ and } q<n\} \) is a Gaussian white noise with mean equals to zero and unit variance. Furthermore, \( u_{jt} \perp u_{jt-k} \) for all values of \( j, t \) and \( k \neq 0 \), and \( u_{jt} \perp u_{st-k} \) for all values of \( k \) and \( s \neq j \).

1.2 The stochastic process \( \{\varepsilon^i_t; t \in \mathbb{N} \text{ and } i = 1, \ldots, n\} \) is stationary with mean zero and \( \varepsilon^i_t \perp u_{jt-k} \) for all values of \( i, j, t \) and \( k \).

1.3 \( A^i(L) = [b^i_1(L), \ldots, b^i_q(L)] \) is \( nxq \) matrix whose \( n \) column is given by \( b^i_q(L) = \sum_{k=-\infty}^{\infty} b^i_{qk} L^k \). The filters \( b^i_q(L) = \sum_{k=-\infty}^{\infty} b^i_{qk} L^k \) are square sumable, that is \( \sum_{k=-\infty}^{\infty} (b^i_{qk})^2 < \infty \).

The hypothesis assumed imply that the observable vector \( \{x^i_t; t \in \mathbb{N} \text{ and } i = 1, \ldots, n\} \) is stationary with mean zero for all \( n \). Let \( \sum^{(n)}(\theta) \) be the spectral density matrix to the vector \( x^i_t \). Another consequence of Assumption (1) is the possibility to write \( \sum^{(n)}(\theta) \) as the sum of the spectral density matrix of common component, \( \sum^{(n)}(\theta) \), and the spectral density matrix of idiosyncratic component, \( \sum^{(n)}(\theta) \).

What makes this model different of previous factor models is the possibility to handle the dynamics of large cross-section units. Moreover, the model does not require the orthogonality between the idiosyncratic components. As a consequence, the model requests additional assumptions in order to identify the latent variables above defined.

Assumption 2:

2.1 Let \( \sigma_j(\theta) \) be the element of the \( i^{th} \) row and \( j^{th} \) column of the matrix \( \sum^{(n)}(\theta) \). For all \( i \in \subseteq, \) there exist one real \( c_i > 0 \) such that \( \sigma_j(\theta) \leq c_i \) for \( \theta \in [-\pi, \pi] \).

2.2 The first dynamic eigenvalue\(^5\) of \( \sum^{(n)}(\theta), \lambda_{n1}(\theta) \), is uniformly bounded, that is there exist one real \( \Delta \) such that

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\(^5\) Following the literature, we refer to the latent root and latent vector associated to the dynamics captured by spectral density matrix of any vector. This concept is an extension of well known latent root and latent vector of auto covariance matrix usually used in the static principal components analysis. See Brillinger (1981) for a detailed discussion.
\[ \lambda_{n1}(\theta) \leq \Delta \text{ for all } n \in \mathbb{N} \text{ and } \theta \in [-\pi, \pi]. \]

Let be assumed also that the first q dynamic eigenvalues of \( \sum_{\chi}^{(n)}(\theta) \) diverge almost everywhere in \([-\pi, \pi]\).

As exposed by the authors, there is some intuition behind these hypotheses. For example, the assumption about the bound to the dynamic eigenvalues of the idiosyncratic components’ spectral density seems to indicate that idiosyncratic causes of variance have their effects concentrated on a limited number of observational units, though it is shared by a large number of them. These idiosyncratic causes tend to zero when the number of observational units tends to infinite. On the other side, the divergence in the spectral density matrix of common components seems to imply that the common causes of variation are present in a large number of observational units with non-decreasing importance among them [Forni, Hallin, Lippi and Reichlin (2000): 542].

Provide the assumptions (1) and (2) are fulfilled, the authors propose the first result about the representation of model:

**Result 1:** The first q eigenvalues of \( \sum_{\chi}^{(n)}(\theta) \) diverge when n tend to infinite almost everywhere in \([-\pi, \pi]\) while the (n-q) rest of them are uniformly bounded.

The proof can be checked in the original paper. It is important to note that the proposition above makes a link between the hypotheses assumed on the set of unobservable variables and the observable properties of the variables under study. Forni and Lippi (1999) show that, under the conditions of result (1), it is possible to write the set of observable variables as the dynamic factor model (1) and (2).

After the considerations about the representation, we may now examine the estimation of the common and idiosyncratic components of the model. Through an application of the Law of Large Numbers, Forni, Hallin, Lippi and Reichlin (2000) show that is possible to estimate the common components projecting the variables \( x^i_t \) in any q linear combinations “properly chosen”. The relevant question now is how one may determine such aggregations. In this same work, the authors suggest the use of the first q dynamic principal components\(^6\) associated to the first q dynamic eigenvalues of the vector \( x^i_t \).

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\(^6\) See Brillinger (1981) for a discussion about principal components.
The dynamic principal components are an extension to the frequency domain of the well-known static model. Let \( P_j(\theta), \theta \in [-\pi, \pi], \) be the dynamic eigenvector associated to \( j^{th} \) eigenvalue of the spectral density of \( x_t^i \). The Dynamic Principal Components are the \( q \) stochastic processes formed by the projection:

\[
Z_{jt} = P_j(L)x_t^i
\]  

(3)

Where \( j = 1, \ldots, q \). Following the notation above, the estimator proposed by Forni, Hallin, Lippi and Reichlin (2000) for the common component vector may be represented by the following projection:

\[
\chi_t^i = [p_{1,i}(L)P_1(L) + p_{2,i}(L)P_2(L) + \ldots + p_{q,i}(L)P_q(L)] x_t^i = K_i(L)x_t^i
\]  

(4)

Where the \( k^{th} \) coefficient of filter is \( p_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_j(\theta)e^{k\theta} d\theta \). Indexing in \( n \) the sample equivalent to the projection above, we can present the important result, proved by Forni, Hallin, Lippi and Reichlin (2000).

**Result 2:** \( \lim_{n \to \infty} \chi_{t,n}^i = \chi_t^i \) in mean square for all \( i \) and \( t \).

Details about the sample properties of the estimators as well as the convergence rate required for the cross-section and temporal dimension can be found in Forni, Hallin, Lippi and Reichlin (2000) and (2004).

An aspect not still mentioned is the determination of the number of common factors. The dimension of vector \( u_t \) can be understood as the fundamental dimension for the dynamic of \( x_t^i \). Unfortunately, there is no well-established formal test in the literature\(^7\). Forni, Hallin, Lippi and Reichlin (2000) propose an heuristic procedure based on result (1).

The spectral density matrix of \( x_t^i \) can be decomposed in terms of its dynamic eigenvalues and eigenvectors:

\[
\Sigma^{(n)}(\theta) = P(\theta)\Lambda(\theta)P(\theta)'
\]  

(5)

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\(^7\) Bai and Ng (2002) propose a procedure to determine the number of factors in the static model.
Where $\Lambda(\theta)$ is a diagonal matrix with the dynamic eigenvalues, classified by their magnitude, from the higher to the lower value, for each frequency $\theta \in [-\pi, \pi]$. The $P(\theta)$ contains the eigenvectors associated with each eigenvalue.

The rank of $\sum^{(n)}(\theta)$ and the number of common factors, $q$, is equal to the number of eigenvalues not equal to zero for each frequency. The dimension of the common shock vector, $u_t$, can be determined by the number of dynamic eigenvalues required to explain the greatest part of $\sum^{(n)}(\theta)$ trace, for each frequency $\theta$. It consists in verifying the number of dynamic eigenvalues of the vector $x^i_t$ that are different from zero on the range of the frequencies $\theta \in [-\pi, \pi]$.

It is clarifying to emphasize that the model above can be understood within the multivariate time series theory, as shown in Brillinger (1981). As defined by expression (1) and supposing that the closest the common components are to the observable series the better, the problem can be stated as the following minimization problem:

$$\epsilon^i_t = x^i_t - \chi^i_t$$

(6)

Where the common components are described as in the expression (1). Let $B^i(L)$ be a matrix of rational functions in the lag operator $L$, which has a reduced rank, taking the information from a $n$-dimensional initial vector to a $q$-dimensional vector, such that $q<<n$:

$$u_t = B^i(L)x^i_t$$

(7)

This allows the model (1) to be written as:

$$x^i_t = A^i(L)B^i(L)x^i_t + \epsilon^i_t$$

(8)

Consequently, in order to minimize the expression (6), we have to properly choose the filters $A^i(L)$ and $B^i(L)$ that make the linear combination $A^i(L)B^i(L)x^i_t$ the nearest to $x^i_t$. Brillinger (1981) has proved that the solution of this problem is obtained by the projection of the $q$ eigenvectors associated to the $q$-first spectral density’s eigenvalues of $x^i_t$. That is, the estimation procedure proposed by Forni, Hallin, Lippi and Reichlin (2000) for the common component produces the best approximation for the series $x^i_t$. Furthermore, the minimum value for the idiosyncratic
components is given by \( \int_0^{2\pi} \sum_{j>q} \lambda'(\theta) d\theta \). So, the closest to zero the last \((n-q)\) dynamic eigenvalues are, referred in result (1), the better the common component is identified.

3. Main Results

We use monthly observations from the Morgan Stanley Composite Index (MSCI) for the period of December 1992 to May 2004. Our sample is composed of the following indexes: Argentina (MSAR index), Brazil (MXBR Index), Mexico (MXMX Index), South Africa (MSE-USSA Index), India (MSEUSIA Index), Indonesia (MSEUSINF Index), South Korea (MSEUSKO Index), Malaysia (MSDUMAF Index), Taiwan (MSEUSTW Index), Israel (MXIL Index), Turkey (MSEUSTK Index), China (MSEUSCF Index), USA (Dow Jones, INDU Index), EUA (NASDAQ, CCMP Index), EUA (Standard & Poor’s, SPX Index), Germany (DAX Index), United Kingdom (FTSE, UKX Index), Japan (Niki, NKY Index), Japan (Topix, TPX Index), Hong Kong (HSI Index), France (CAC Index), Australia (AS51 Index), Spain (IBEX Index) and, finally, Italy (MIB30 Index).

The methodology makes use of the estimation of spectral densities, what requires stationary variables. In order to ensure stationarity, we take the first difference from the logarithm of all series.

As a first step to apply the dynamic factor model, we need to determine the number of common shocks to the set of series under study. Forni, Hallin, Lippi and Reichlin (2000) proposed two distinct methods to distinguish the number of common factors\(^8\): the graphic below shows the percentage of variance explained by each of the dynamic eigenvalues. Forni, Hallin, Lippi and Reichlin (2000) and Favero, Marcellino and Neglia (2002) state that there must exist a significantly large gap between the percentage of variance of \(x_i\) explained by the first \(q\) dynamic eigenvalues and the one explained by the \(q-th + 1\) one.

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\(^8\) All estimations were performed with Matlab version 6.0 and the codes made available by Forni, Hallin, Lippi and Reichlin (2000) at www.dynfactors.org.
As we can observe, the first dynamic eigenvalue explains almost half of the variance of $x_t$. The second one, on the other hand, explains considerably less, about 10% of the variance. From the third eigenvalue on, the marginal contributions are all below 5%.

The second criteria for the determination of the number of factor suggested by Forni, Hallin, Lippi and Reichlin (2000) is: (i) to recursively estimate the spectral density matrix of a sub-set of $x_t$; (ii) to calculate the dynamic eigenvalues; (iii) to choose the number of common factors, $q$, on the following rule: as the number of series, and consequently, of the eigenvalues grow, the average of the first $q$ dynamic eigenvalues diverge, meanwhile the average of the $q + 1$ dynamic eigenvalue keeps relatively constant.

The graphic below shows the average of the eigenvalues as the series are added to the model. As we can see, the average of the first eigenvalue diverge at a much higher velocity then the average of the other dynamic eigenvalues. Additionally, the average of the following two dynamic eigenvalues distinguish themselves from the averages of the others eigenvalues.
Following the evidence above, the model was estimated with only three factors, given the low contribution from the other factors to the common variance of the series.

In the appendix, we report the peridiogram for the estimated common components and the peridiogram for all stock markets. For illustration, the graphic below reports the respective peridiogram for Brazil, which brings some interesting information.

The peridiogram is constructed on the basis of a decomposition of the spectral density of the series, showing how its variance distributes itself between the cycles of different frequencies. The graphic has, in the $x$-axes, the frequency of the cycles normalized to $\pi$ and, in the $y$-axes, the total variance. As it can be seen, the original series has its variance explained basically by bi-monthly cycles (frequency $0.9\pi$) and semester cycles (frequency $0.3\pi$). Looking for the common component of the Brazilian index, it can be noted that its variance is composed mostly by semester cycles.
The same pattern seems to repeat itself with the remaining indexes from developing countries. With the exception of Malaysia, in all other cases, the common component has its variance explained by longer cycles than the original series does. In contrast, in the case of developed nations, the original series and their respective common components are explained by cycles of equivalent frequencies.

The result above helps us to understand the relative success of cointegration analysis to the study of the integration of developing countries stock markets. On the other hand, in the developed countries case, it seems necessary to consider higher frequencies of the spectrum.

The table below illustrates the variance of the series due to the common component. The first result that distinguishes itself is the high percentage of the dynamics of the series owing to the common component. As the table shows, on average, almost 67% of the variance from the whole set may be attributed to the common shocks. The high share of the common component in the total variance of the indexes seems to be evidence of a high integration of the stock markets worldwide. More than merely providing evidence of co-movement between the markets, the result below supports the idea that forces that are common to their individual dynamics drive the world stock markets.
It is important to highlight that the fraction of the total variance caused by the common component varies significantly between countries. However, some patterns can be recognized. The percentage of the variance explained by the common component seems to be higher between the stock exchanges from developed nations, as in the case of the Dow Jones, Nasdaq, S&P, FTSE, Dax, CAC, Nikkey, Topix and Madrid. In all those cases, that percentage is higher than 70%. Alternatively, in developing nations as Brazil, Mexico, South Korea, South Africa, Hong Kong and China, this stake is slightly lower, between 50% and 60%. Finally, we have the stock exchanges whose variance is determined primarily by idiosyncratic shocks, as Argentina, India and Turkey.

These results imply that risk diversification is indeed an option to international investors. Distributing a portfolio between the three groups above may indeed decrease the investors’ exposure to global common shocks, since the groups have their dynamics driven by significantly different proportions of common shocks.
4. Conclusions

Making use of a dynamic factor model, the present paper has the objective of proposing a methodology to infer the level of dynamic integration for a set of stock markets. The model permits the estimation of the common component, itself the result from common shocks affecting all markets, from each of the stock exchange under study.

Two conclusions emerge: first, the results reveal that the variance caused by the common component varies significantly between countries. Notwithstanding, we are able to infer from the data that the level of dynamic integration has a positive correlation with the country’s overall development, with developed nations experiencing a higher level of integration than developing nations. Consequently, international portfolio diversification is indeed possible between stocks of developed and developing countries stock markets.

Second, it is also very interesting to note that the peridogram of developing countries’ stock exchanges indicate that the common component of their respective markets are driven, predominantly, by shocks of lower frequency. In contrast, the peridogram of developed countries’ stock exchanges are considerably influenced by fluctuations of higher frequencies. That pattern may be responsible for the difficulty in finding common stochastic trends between developed and developing markets. The Dynamic Factor Model proposed makes clear distinction between co-movement at all phases of the spectrum, what allows us a better identification of the integration of international stock markets.
REFERENCES


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Appendix

Original Series and Common Components Estimated: Series and Spectral Density

Figure B.1 – Argentina

Figure B.2 – Mexico

Figure B.3 – South Africa
Figure B.4 – India
Figure B.5 – Indonesia
Figure B.6 – South Korea
Figure B.7 – Malaysia
Figure B.12 – Dow Jones

Figure B.13 – NASDAQ

Figure B.14 – S&P 500

Figure B.15 – Frankfurt Dax
Figure B.16 – London FTSE

Figure B.17 – Tokyo Nikkey

Figure A.18 – Tokyo Topix

Figure B.19 – Hong Kong