Inventories and Capacity Utilization in General Equilibrium*

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Abstract

This paper examines the role of inventories and capacity utilization (of both capital and labor) and, in particular, the relationship among them in the context of a dynamic stochastic general equilibrium framework. The analysis is centered on the effects of two possible shocks: preference (demand) shocks and technology shocks. Impulse-response experiments show that inventories and the rate of capital utilization are mostly complements, while inventories and the rate of labor utilization are mostly substitutes. Moreover, low-persistence shocks emphasize the role of inventories as being a “shock absorber”, whereas high-persistence shocks emphasize the role of inventories as being a complement to consumption. Consistent with the stylized facts in the literature, simulation results show that inventory holdings are procyclical, while the inventory-to-sales ratio is countercyclical. An additional “theme” regarding the treatment of uncertainty is explored. By approximating the decision rules to a second order, we observe that higher exogenous uncertainty enhances the importance of the precautionary motive to holding inventories.

Keywords: Inventories, capital utilization, labor intensity, business cycle.

JEL Classification: E22, E32, J22.

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1 Motivation

Previous studies have shown that inventory fluctuations play an important role in explaining the business cycle. Although inventory investment has averaged roughly one-half of 1 percent of U.S.' GDP over the post-World War II period, changes in inventory investment have averaged more than one-third the changes in quarterly GDP (see Fitzgerald, 1997). Moreover, Blinder and Maccini (1991) show that the drop in inventory investment accounted for 87% of the drop in total output during the average postwar recession period in the U.S. In words of Blinder, “business cycles are, to a surprisingly large degree, inventory cycles.” (Blinder, 1990, p. 8).

The same way, capacity – or factor – utilization, being a central component of the economy’s supply side, is often considered as an indicator of the state of real activity. For instance, variable factor utilization is thought to account for much of the variation in the Solow residual (40-60 percent according to Basu and Kimball, 1997), and thus provide an important insight into characterizing the business cycle.

The primary goal of this work is to characterize the relationship between inventory investment and capacity utilization – i.e., the intensity of use of both capital and labor – in the context of a general equilibrium Walrasian stochastic framework. Likewise, I am interested in answering the question of whether these variables respond symmetrically to shocks that may differ in their rates of persistence.

The subject associated with understanding the relationship between capacity utilization and inventory investment relies on the fact that inventories are in some way an alternative form of capital, and capital, as well, is somehow a form of inventory. Once the possibility of variable rates of utilization of both capital and labor is also considered, then both such rates of utilization and inventories can be conceived as providing a short-run adjustment “buffer stock” mechanism.

Christiano (1988) pictured inventories as a form of capital as well, treating them as an investment component that becomes a residual during the business cycle. In a broader sense, inventories play the role of “buffering consumption from unexpected disturbances in production, and buffering production from unexpected disturbances in consumption.” (Christiano, 1988, p. 248.) This attribute of inventories is what distinguishes them from other types of capital indeed; all those types involving rigidities to some extent.

This work develops a dynamic, stochastic general equilibrium (DSGE) model that departs from standard models in that it introduces endogenous capital depreciation, variable use of the labor force, and inventory holdings.

Endogenous capital depreciation captures the idea that the depreciation rate is a variable subject to choice by the user of the capital good. As such, this rate enters into the
model as a function of the intensity of use of capital (see, e.g., the works of Greenwood, Hercowitz, and Huffman, 1988, on variable capital utilization in a real business cycle model, and of Rumbos and Auernheimer, 2001, on variable capital depreciation in an otherwise standard neoclassical growth model).

Variable labor utilization is introduced into the model by allowing the agent to modify the working time during the production process. Within the macroeconomic literature on inventories, for instance, Galeotti, Maccini, and Schiantarelli (2005) also distinguish between an employment decision (extensive margin), and an hours-per-worker decision (intensive margin). Following the “factor-hoarding” literature (e.g., Burnside, Eichenbaum, and Rebelo, 1993, and Burnside and Eichenbaum, 1996), I will assume that the demand for productive factors is decided before the state of nature is known, while capacity utilization is decided later with full information.

Finally, the demand for the zero-return assets is motivated here by the idea that a larger stock of inventories allows consumers either to match their tastes more effectively or to economize on shopping costs. Technically, this is done by introducing inventories in the utility function, as is done by Kahn, McConnell, and Perez-Quiros (2002). This approach is different to the widely-used approach of introducing inventories as inputs in the production function (see, e.g., Kydland and Prescott, 1982). Although the focus of this work is on the analysis of final good inventories, one can still apply this argument to the analysis of intermediate good inventories. For instance, one can think of producers with larger inventories of intermediate goods as ultimately increasing consumers’ possibilities – and utility – by expanding the different alternatives.

Regarding the uncertainty involved in this economy, the model features preference and technology shocks. These shocks are important for the analysis as they give rise to another motive for the agent to hold inventories. This motivation is linked to the “stock-out avoidance” motive – see, e.g., Kahn (1987) – in that the agent shields herself from the uncertainty associated to the disentanglement between the commitment of factor inputs and the realization of the shocks.¹

Results based on impulse-response analysis show that low-persistence shocks to preferences that relatively raises utility on both consumption and inventories make the agent reduce transitorily inventory holdings. On the other hand, these shocks lead to an increase in the intensity of use of both capital and labor. This is an appealing result one expects to hold if inventories are interpreted as a buffer for consumption. In contrast, both low-persistence shocks to technology and high-persistence shocks either to preferences or technology make inventories complement both rates of utilization. In other words, the

¹See also Khan and Thomas (2007b) for a thorough discussion on the pros and cons of stockout-avoidance type-of models.
latter emphasize the role of inventories as being a complement to consumption instead of emphasizing their role as a “shock absorber”.

The model is successful in accounting for the average level of inventories in the U.S. economy and in replicating the business-cycle facts on inventories and capacity utilization. In particular, the inventory-to-sales ratio is countercyclical, while inventory investment is procyclical.\footnote{See also Ramey and West (1999) and Wen (2005) regarding the cyclical regularities of inventories.}

On the robustness of the model, I discuss an alternative way of modeling variable capital utilization proposed by Christiano, Eichenbaum, and Evans (2005). Instead of assuming an endogenous rate of depreciation, they simply impose a cost in terms of output of using capital more intensively. We will notice that, by taking into account this alternative, the qualitative features remain unaltered, though quantitative results are indeed affected.

An additional issue is investigated in this work. It has to do with the treatment of uncertainty and the consequences of using, as it is done in most of the literature, a first-order approximation to the decision rules. By approximating to a second order, one observes – as expected – that higher exogenous uncertainty enhances the importance of the precautionary motive to holding inventories.

The paper proceeds as follows. The next section presents the related literature. Section 3 introduces the general equilibrium model which later on is referred to as the benchmark specification. In that section, I describe the environment, discuss the assumptions of the model, and provide the optimality conditions for the benchmark setting. Section 4 proceeds with the solution to the model, and shows the main results. Section 5 provides an analysis of the robustness of the benchmark model. Finally, Section 6 presents the conclusion to this work.

2 Related Literature

Fitzgerald (1997, p. 12) summarizes the macroeconomic literature on inventories by stating that “this literature provides a good example of how theory and data interact in the ongoing process of research.” The main question within this area of research has been whether inventory investment was in fact a key determinant in the amplification and propagation of shocks. Motivated by this inquiry, the classic paper by Metzler (1941) developed a model where exogenous, uncorrelated shocks, together with inventory investment, could generate the business cycle fluctuations of output observed in the data. Later, Holt, Modigliani, Muth, and Simon (1960) introduced, in a simple linear-quadratic model of optimal inventory behavior, the “production smoothing” motive for holding inventories. They assumed convex
costs of production and firms facing variable and uncertain demand for storable goods. In this approach, a profit-maximizing firm will have incentives to hold inventories to smooth the time path of production – inventories act as a buffer stock in production, their mostly recognized role indeed.

Notice that the production-smoothing notion implies two testable implications: the variance of sales exceeds the variance of production, and inventories and output are negatively correlated. The idea is simple. Since the cost of production is convex, firms will generate a surplus when sales are low, and use this surplus when sales are high. Providing microeconomic foundations to the business-cycle analysis of inventory behavior, this was considered the standard theory until the early 1980s.

Works by Blinder (1981, 1986) and Blanchard (1983), among others, put in evidence the weaknesses of the standard model in replicating some of the stylized facts on inventory cycles. These facts are: (i) at the aggregate level, output is more variable than sales, and (ii) inventory investment is procyclical. In order to explain these contradictions, the new class of models basically added shocks to the firm’s production costs. At the same time, though focused on a different motivation, Kydland and Prescott (1982) developed a model where inventories enter the production function, and uncertainty arises from productivity shocks. They found that both cyclical fluctuations in inventories and their correlation with output were consistent with the data.

On the other hand, Kahn (1987) provided a theoretical model where firms followed a target on inventories, introducing a stock-out avoidance motive for inventory investment. Under the assumption that production takes time, firms demand inventories so as to be ready for unexpected increases in sales. The paper rationalizes this motive by introducing both a non-negativity constraint on inventories and serially-correlated demand shocks. These two assumptions generate that the variance of production exceeded that of sales. More recently, Bils and Kahn (2000) developed a model where final good inventories facilitate sales, and a stockout-avoidance motive dominates the rationale for holding these stocks. In line with the evidence, their model generates the result that inventories vary in proportion to expected sales.

In order to eliminate the production-smoothing result, other approaches assume simply that the marginal cost of production is decreasing. For instance, Ramey (1991) found evidence of non-convex marginal costs in a large number of manufacturing industries. While most of this literature has concentrated in modifying the production-smoothing model, an alternative theory, the so-called \((S,s)\) approach, focused on the timing of deliveries rather than the timing of production (e.g., Caballero and Engel, 1999, Fisher and Hornstein, 2000, and the recent contributions by Khan and Thomas, 2007a,b).

Wen (2005) concludes that the stock-out avoidance approach seems to have better po-
tential than other theories for explaining inventory fluctuations. Distinguishing between low and high-frequency time series, he found that this approach performs well at different cyclical frequencies and, most importantly, that demand shocks are the main source of the business cycle.

Recently, Galeotti et al. (2005) developed a stockout-avoidance type-of model integrating inventory and labor decisions, distinguishing between an extensive and an intensive margin decision. Their model is closely related to this work, in particular, regarding the time structure of information and decisions. In that paper, the firm decides on inventory investment and employment (the extensive margin) before sales and technology shocks are known. Later on, the firm adjusts the intensity of use of labor (the intensive margin). The main differences between Galeotti et al. (2005) and this work lie on that they use a partial equilibrium approach, and capital accumulation is absent. In contrast, this paper integrates inventory investment with both variable capital and labor utilization in a general equilibrium environment.

Iacoviello, Schiantarelli, and Schuh (2007) is another recent approach closely related to this work. In a DSGE model, the firm decides on both inventory investment and capital utilization, while inventories provide utility to the consumer. Main differences between this paper and their’s include the time structure of decisions and information, the absence of variable use of labor in their model, and the fact that they emphasize the roles of both intermediate and final good inventories.

Regarding the literature on variable capacity utilization and, in particular, endogenous depreciation, Calvo (1975) and Greenwood et al. (1988) are standard references – although one could go back, for instance, to Keynes’ notions of “user cost”. Greenwood et al. (1988) introduce variable capacity utilization into a standard, real business cycle model, where shocks to investment generate business fluctuations. In both their model and this paper, variable capacity utilization allows predicting the Keynesian type result of less than full-capacity equilibrium. Nevertheless, Greenwood et al. (1988) consider only variable capital utilization, excluding variable labor intensity from the analysis.

Burnside and Eichenbaum (1996) studied the capacity-utilization issue introducing variable rates of utilization of both capital and labor. They analyze the role of these rates in propagating shocks over the business cycle in a model where capacity utilization is treated as a form of factor-hoarding. As in Greenwood et al. (1988), they assume that the rate at which capital depreciates is a function of the capital-utilization rate.

More recently, Rumbos and Auernheimer (2001) introduce variable capital utilization into a modified Ramsey-type model by modeling the notion of pure user cost. They find that the introduction of a variable utilization rate yields a slower rate of convergence toward the steady state, inducing more persistence in the transitional dynamics.
Variable capacity utilization, and especially variable capital utilization, is now being widely used in standard, DSGE models in many areas of macroeconomics. For instance, Christiano et al. (2005) introduce capital utilization in a study of nominal rigidities and monetary policy. Not surprisingly, one of the key features of their model that serve to account for the observed inertia in inflation and persistence in output is variable capital utilization. An interesting feature of their model economy is that it departs from the standard specification that assumes endogenous capital depreciation. They assume constant depreciation in a model where varying the capital-utilization rate implies a cost that is borne through less resources in consumption goods. In Section 5, I will go over this particular issue in order to compare how different the conclusions are if one took that approach.

3 A General Equilibrium Model

Consider a standard, dynamic stochastic general equilibrium model, modified to incorporate inventory holdings and variable capacity utilization of both capital and labor. Inventory holdings are introduced into the model by assuming that they provide utility services as a proxy for decreasing shopping time and increasing variety. Being an asset, they also shield the agent from the uncertainty arising on the timing gap between production decisions and consumption and investment decisions. Variable capacity utilization is introduced into the model by allowing the agent to choose the rates at which capital and labor are used during the production process. Capital utilization will involve an endogenous capital depreciation rate, while labor intensity will involve an additional disutility term to the agent arising from adjustments on work time.

3.1 The Environment

The model economy is populated by a continuum of identical agents with unit mass. There is one good that may take the form of consumption, capital, and inventories.\textsuperscript{3} Output is produced according to a production function $Y_t = F(e_t N_t, s_t K_t; \omega_t)$; where $N_t$ and $K_t$ denote the levels of labor and capital at time $t$, $e_t$ and $s_t$ denote the rates of labor and capital utilization at time $t$, and $\omega_t$ represents the exogenous, stochastic time-$t$ level of technology. As stated by this production function, what matters for producing output is the total amount of effective capital, $s_t K_t$, i.e., capital services per unit of time, and

\textsuperscript{3}Inventories are final-good inventories in this one-sector model. Other papers analyze both final and intermediate-good inventories, highlighting the importance of separating them. The main reason for studying both categories appears from the fact that, as the data show, intermediate-good inventories are relevant both as a fraction of total inventories (they are roughly more than one-third of total inventory holdings) and in terms of their relative volatility (they roughly represent almost one-third the variability of total inventories). For evidence on this issue, see, e.g., Blinder and Maccini (1991).
effective time of work, \( e_t N_t \), i.e., labor services per unit of time. I will assume that markets are competitive in this economy. Given this assumption, the presence of firms will no play any relevant role, thus for now I assume that there are no firms. In Appendix A, I show that the same results arise in a model with competitive firms.

As is standard in the capital utilization literature, the model assumes that using capital more intensively raises the rate at which it depreciates. For that matter, I will assume that the time-\( t \) depreciation rate of capital, \( \delta_t \), is an increasing and convex function of \( s_t \), given by \( \delta_t = \delta(s_t) \). Therefore, the stock of capital evolves according to:

\[
K_{t+1} = K_t (1 - \delta_t) + I_t,
\]

where \( I_t \) denotes time-\( t \) gross investment in physical capital.

At any date \( t \), the resource constraint of the representative agent is given by:

\[
C_t + I_t [1 + h(I_t)] + Q_{t+1} - Q_t \leq Y_t,
\]

where \( C_t \) denotes time-\( t \) households consumption, \( Q_t \) denotes inventory holdings at time \( t \), and \( h(I_t) \) is an increasing function that captures adjustment costs on investment, such that the total investment cost, \( I_t h(I_t) \), is convex.\(^4\)

The typical agent in this economy maximizes her expected lifetime utility as given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, l_t, e_t, Q_t; z_t),
\]

where \( 0 < \beta < 1 \) is the discount factor, \( l_t \equiv 1 - e_t N_t \) stands for leisure at time \( t \) − out of a normalized time endowment of 1 −, and \( z_t \) represents an exogenous, stochastic preference shock.

The reason why the third argument, \( e_t \), and the fourth argument, \( Q_t \), are listed in \( u \) plays an important role, so it warrants some discussion. On the third element, the rate of labor intensity, it is clear that it affects the agent’s utility through its effect on leisure, since the time-\( t \) leisure is defined as \( 1 - e_t N_t \). However, by introducing a separate argument for \( e_t \) one may clearly understand the effects of allowing the agent to independently choose the intensity at which the leisure time is used. This will become clear below, once the assumptions made on the information structure are described. Moreover, the time-\( t \) instantaneous utility function further includes the stock of inventories at time \( t \). Its motivation was briefly

\(^4\) Notice, first, that adjustment costs are not defined as a function of gross investment relative to the stock of capital. Second, notice that inventories do not depreciate (other studies introduce a fixed depreciation rate in order to capture inventory holding costs). Nevertheless, these simplifications are not central for the questions I intend to address.
mentioned in the introduction, and it will become clearer later when studying its effects. The way they enter in \( u \) is consistent with others settings in the inventory literature (e.g., Iacoviello et al., 2007, and Kahn et al., 2002).

### 3.2 Time Structure of Decisions and Information

At the beginning of date \( t \), before the state of nature is known, the agent must decide on the time she will allocate to work at time \( t, N_t \). The stock of capital available for production at time \( t, K_t \), is given by the investment decision at the previous date. Thus, at the moment the shocks are finally revealed, the stocks of factors of production available for producing output are given. These assumptions, and in particular the assumption related to the labor decision, indicate that the model is, somehow, a stock-out avoidance kind-of model. Remember that, in those models, the time structure of decisions and information plays an important role. That is, the fact that the stocks of factors of production are given at the moment the demand and the state of technology are revealed, gives rise to another motive for holding inventories (in addition to that of saving shopping time), which is that of protecting the agent from uncertainty.

After the shocks are revealed, the agent chooses the rates of both capital utilization and labor intensity, \( s_t \) and \( e_t \), in order to adjust the services of capital and labor to the new information. Finally, the agent decides how much to consume, \( C_t \), and how much to invest for the next period both in physical capital, \( K_{t+1} \), and in inventory holdings, \( Q_{t+1} \). See the timeline in Figure 1.\(^5\)

### 3.3 A Discussion on the Rate of Labor Intensity

Having described the information structure, it remains to explain the appearance of \( e_t \) as a separate element in the utility function. The argument goes as follows. Once the agent has already decided on the time devoted to work at \( t, N_t \), the best she can do in response to a shock is to vary this pre-committed working time by choosing the appropriate rate of labor intensity. However, one should expect that, as the agent moves away from the previously optimal time to work, it does not come for free. In other words, it makes sense that the agent faces a cost in terms of disutility of changing the plans already made. This cost is captured here by identifying a separate argument that is additional to the "linear" effect of \( e_t \) on leisure, as measured by \( l_t \). \textit{Ex-ante}, whether the agent increases \( e_t \) or \( N_t \) will be the same in terms of disutility arising from effective work. \textit{Ex-post}, changing \( e_t \) will not be the same as changing \( N_t \) for it will involve an extra-cost in terms of disutility – despite the fact that \( N_t \) cannot be changed after the shock indeed.

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\(^5\)Figures and tables are shown in Appendix C.
Consider, as an example, the difference between extensive and intensive-margin decisions related to the demand for labor by the firm. The extensive margin refers to the firm’s demand in terms of the number of employees – translated to this model as the amount of labor that the agent decides before the shocks are realized, $N_t$. The intensive margin refers to the firm’s decision on the number of hours the workers will be effectively working, $e_t N_t$, which can differ from that previously established in labor contracts. That is translated to this work as the decision on the effective labor, which ultimately depends on the rate of labor utilization.

Both in theory and in practice, it is well known that a decision on the intensive margin that implies a change in the number of hours established in labor contracts, say extra hours or lay-offs, in general involves an extra cost for the firm. Extra hours are paid more than the average hourly wage rate, while lay-off hours, even though not productive, still involve some costs for the firm. The implications of a specific market structure that can support these considerations and, more in general, can support the representative-agent model described in this section will be carefully discussed in Appendix A.

One way of rationalizing the last argument under a representative agent problem implies thinking on adjustment costs in the form of disutility. The motivation can briefly be stated as follows. Consider an individual who has already planned her day regarding the time allocated to both work and leisure. Further, suppose this individual made plans already for the time immediately after work. Suppose now that all of a sudden she receives an assignment at the very last minute that requires her to stay one more hour. What happens is that this individual has to make changes in her plans, which ultimately imply an extra cost in addition to the cost of working the extra hour. This extra cost on changing plans would not exist if the individual had originally planned on staying one more hour at work. What happens is that this individual has to make changes in her plans, which ultimately imply an extra cost in addition to the cost of working the extra hour. This extra cost on changing plans would not exist if the individual had originally planned on staying one more hour at work. The same occurs if this individual receives the news that she can leave the workplace one hour earlier. It is clear that this person is happier now because of this extra time for leisure, but one should expect this person would have been happier if she could originally make plans for that extra hour of leisure. It is this extra cost what led me to introduce the rate of labor utilization as a separate argument in utility, which I believe will also ease the exposition later on.

3.4 Dimensions of the Rates of Utilization

I have defined time-$t$ labor services and capital services as $e_t N_t$ and $s_t K_t$, respectively. For ulterior use, let me now set $L_t \equiv e_t N_t$ and $S_t \equiv s_t K_t$. These are the flows of factor services per unit of time that are used to produce output with the technology given by $F(L_t, S_t)$ – I abstract for now from the level of technology, $\omega_t$. 9
Variables $e_t$ and $s_t$ represent the intensities of use of labor and capital. In the case of the former, it can simply be interpreted as an index of the period-$t$ utilization rate of the stock of labor, which in turns is measured here in fractions of a unit of time. In the case of the latter, $s_t$ can be interpreted in two ways: (i) it can be understood as measuring the speed at which the stock of capital is operated per unit of time, with a given amount of labor services, or (ii) it can be understood as measuring the fraction of a unit of time at which the stock of capital is operated, given a capital-labor services ratio. To put this notion formally, consider the following argument based on Calvo (1975) and Burnside and Eichenbaum (1996). Define $\overline{eN}(K, sK/eN)$ as the flow of labor services required to operate capital at the maximum speed, when $S \equiv K$, with the capital-labor services ratio equal to $sK/eN$. This ratio can be interpreted as the amount of capital at the disposal of the typical individual during her shift. This implies that

$$\frac{K}{\overline{eN}(K, sK/eN)} = \frac{sK}{eN};$$

and then, $seN = eN$. Finally, notice that the production function being homogeneous of degree one – which is the case, as it will be assumed later – implies that

$$F(seN, sK) = sF(eN, K).$$

The expression $F(eN, K)$ can be interpreted as the “full-capacity” output when the factor services ratio equals $sK/eN$ and total supply of capital services is $K$. This reasoning is then intending to show that the rate of capital utilization, $s$, can be conceived as the share of the maximum output obtained when “machines” are operated a fraction $s$ of a unit of time, per unit of time, for a given factor services ratio – e.g., $s = 0.2$ implies that capital is being used 20% of time per date.

On the complementarity between the rates of labor and capital utilization, consider also the following reasoning. Assume a firm that uses different combinations of services from computers and workers to produce a unit of output under a given technology. Further, take the set up discussed so far, where capital and labor, i.e., computers and workers, can be utilized at different rates. Now, one might raise the question of whether, say, an increase in the intensity of use of labor would automatically imply an increase in the intensity of use of capital, and viceversa. After all, if we say that the workers are the ones who use the computers, any rise in the intensity of use of labor, suppose an extra-hour, would immediately imply a rise in the intensity of use of computers – the extra-hour – other things the same.

A corollary of the argument made above would be that the two rates of capital utiliza-
tion should be clearly related in a one-to-one basis. However, even when the argument is correctly placed, there is a mistake in the conclusion one is tempted to assert. In fact, we do not need to specify additional, technical assumptions on the relationship between these two variables. Again, consider the extra-hour arising from an increase in the rate of labor utilization. In some way, it does not necessarily mean that computers will also be used one more hour, but simply that individuals will work more for an hour, perhaps with the same or less computer services than before, on average. After all, these workers might be willing to change the technology towards a labor-intensive production process. Of course, if the production function is of a constant-return-to-scale technology, then the two rates will be positively related but, once more, it does not mean that they should have a one-to-one relationship imposed by assumption.

3.5 On the Properties of the Utility Function

A utility function \( u(C_t, l_t, e_t, Q_t; z_t) \) that satisfies the aforementioned properties of \( e_t \) must satisfy the following assumptions: \( u_3(\cdot) \geq 0 \) for \( e_t \leq 1 \), and \( u_{33}(\cdot) < 0 \) for all \( e_t \). These assumptions imply the existence of a cost or “punishment”, in terms of utility, to any deviation of \( e_t \) from its \textit{ex-ante} optimal choice \( \bar{e} \). As shown below, it will be the case that \( \bar{e} = 1 \). In particular, the property \( u_3(e_t = 1, \cdot) = 0 \) comes from the fact that at this point, \( e_tN_t = N_t \), which means that there is no deviation from the decision on leisure already planned, i.e., there are no extra costs in terms of disutility. Only at those points where \( e_t \neq 1 \), the agent can do better by getting closer to the optimal, \textit{ex-ante} time allocation. For a simple illustration of a utility function that satisfies these properties, see Figure 2.

With regards to inventory holdings, I assume that \( u_4 > 0 \) and \( u_{44} < 0 \), for all \( Q_t \). That is, the agent receives utility services that are proportional to inventory holdings, and, as it is commonly assumed for the consumption good as well, the marginal utility on inventories is decreasing. These assumptions are consistent with those considered in Iacoviello et al. (2007) and Kahn et al. (2002).

3.6 The Optimization Problem

Given the assumptions made so far, the competitive equilibrium corresponds to the solution for the aggregate of all representative individuals of this economy. I will discuss now the problem of the typical agent as was described above. The competitive-equilibrium problem with firms is discussed in Appendix A.

To posit this problem in as a stylized way, consider the following notation. Define as \( \Gamma_{t-1} \equiv (\omega_{t-1}, z_{t-1}) \) the set of previous date’s realizations of the exogenous shocks. Then, the time-\( t \) set of state variables for the optimization problem will be given by \( \Phi_t \equiv \)
\{K_t, Q_t, \Gamma_{t-1}\}$. Among the time-\(t\) choice variables is \(N_t\) (decided before the shocks are revealed) and the set \(\Omega_t \equiv \{C_t, e_t, s_t, K_{t+1}, Q_{t+1}\}\), which includes the variables to decide after the new information is received.

In order to present the problem in a way that reflects explicitly the informational constraints, I set up the following dynamic programming problem.\(^6\)

\[
V(\Phi_t) = \max_{N_t} \{ \max_{\Omega_t} \left[ u(C_t, l_t, e_t, Q_t; z_t) + \beta V(\Phi_{t+1}) \right| \Gamma_t \} \}
\]  

subject to (1) and (2), \(K_{t+1} \geq 0, Q_{t+1} \geq 0\), and given \(K_0\) and \(Q_0\).

The solution to this problem is characterized by the following five optimality conditions – in addition to (1), (2) given with equality, and the transversality requirements.\(^7\)

\[
E \{ e_t \times [u_1(C_t, l_t, e_t, Q_t)F_1(e_tN_t, s_tK_t) - u_2(C_t, l_t, e_t, Q_t)] \} \big| \Gamma_{t-1} = 0 \tag{7}
\]

\[
u_1(C_t, l_t, e_t, Q_t)F_1(e_tN_t, s_tK_t) - u_2(C_t, l_t, e_t, Q_t) = -\frac{u_3(C_t, l_t, e_t, Q_t)}{N_t} \tag{8}
\]

\[
F_2(e_tN_t, s_tK_t) = \delta'(s_t)p_{k,t} \tag{9}
\]

\[
u_1(C_t, l_t, e_t, Q_t)p_{k,t} = \beta E \left\{ \begin{array}{c} u_1(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1}) \times \\ s_{t+1}F_2(e_{t+1}N_{t+1}, s_{t+1}K_{t+1}) + \\ (1 - \delta_{t+1})p_{k,t+1} \end{array} \right\} \big| \Gamma_t \tag{10}
\]

\[
u_1(C_t, l_t, e_t, Q_t) = \beta E \left\{ \begin{array}{c} u_1(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1}) + \\ u_4(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1}) \end{array} \right\} \big| \Gamma_t \tag{11}
\]

where \(p_{k,t} = 1 + h(I_t) + I_t h'(I_t)\) can be interpreted as the relative price of a unit of capital good in terms of consumption good. That relative price, which even in steady state will be larger than one, arises from the existence of a cost of transforming a unit of consumption into a unit of capital.

Optimality condition (7) sets the expected marginal product of labor equal to the expected marginal utility of leisure, measured in terms of consumption. This equation is expressed in terms of expectations because, remember, the agent decides labor before the information is known. The second equation (8) refers to the choice of the rate of labor intensity, and it sets the difference between the marginal product of labor and the marginal utility of leisure equal to the marginal disutility of the rate of labor intensity. Notice that if what was planned in (7) is indeed optimal when the shocks are realized, then because of

\(^6\)The way in which I describe the information structure and the timing of decisions is close to the approach followed by Christiano (1988).

\(^7\)To simplify notation, I write both the utility function and the production function without their respective states of preferences and technology, \(z_t\) and \(\omega_t\).
the way I defined $u_3(\cdot)$, it must be the case that $e_t = 1$. This is the same as saying that the right-hand side of (8) is equal to zero, i.e. $u_3(\cdot) = 0$, and thus this condition reduces to (7) \textit{ex-post}. Intuitively, this means that there is no shock that could make the agent deviate optimally from what she had already planned in regards to the allocation of time.

Equation (9) is standard in the literature of variable capital utilization. It characterizes efficient use of capital by stating that the stock of capital should be utilized at the rate $s_t$ which sets the marginal benefit of capital services equal to the marginal user's cost. The latter is made up of two elements. The component $\delta'(s_t)$ is the marginal cost in terms of a higher depreciation from using capital more intensively, whereas the factor $p_{k,t} = 1 + h(I_t) + I_t h'(I_t)$ represents the current replacement cost of old in terms of new capital good.

Equation (10), although a bit “scary”, is a standard optimality condition on investment – an Euler equation. The left-hand side is the loss in current utility from an extra unit invested in physical capital. The right-hand side represents the discounted expected future utility obtained from that unit. The first term inside brackets, $s_{t+1} F_2(e_{t+1} N_{t+1}, s_{t+1} K_{t+1})$, is the marginal benefit in terms of production of a unit of capital in the next period – adjusted for its rate of utilization -, while the second term, $(1 - \delta_{t+1}) p_{k,t+1}$, represents the future value of this unit after depreciation, both margins in terms of consumption.

Finally, equation (11) characterizes the optimal decision of holding inventories. It sets the loss in current utility from an extra unit stored as inventory today equal to the discounted expected future utility obtained from that unit. The second term inside brackets, $u_4(C_{t+1}, L_{t+1}, e_{t+1}, Q_{t+1})$, represents the future utility received from an extra unit of inventory accumulated today, while the first term, $u_1(C_{t+1}, L_{t+1}, e_{t+1}, Q_{t+1})$, represents its future value in terms of consumption, which is indeed the same as the value of an extra unit of consumption good.

\section*{4 Solving the Model: The Main Results}

In this section, I proceed with the solution to the model, and show the main results. First, I discuss the specific functional forms and calibrate the parameters of the model economy. Second, I introduce the analysis of the impulse-response functions and the second-order properties generated by the model. Later on, I present the effects of inventories and capacity utilization, the relationship between them, and the source of shocks behind these results. Finally, I discuss the relevance of considering a stochastic model and, especially, the effect of uncertainty on inventories and capacity utilization.
4.1 Functional Forms and Calibration

The specific production technology is assumed to be

$$F(e_tN_t, s_tK_t; \omega_t) = \omega_t(e_tN_t)^{1-\alpha}(s_tK_t)^\alpha,$$

(12)

where $0 < \alpha < 1$ is the share of income allocated to effective capital.

The state of technology is assumed to evolve according to the following stochastic process

$$\ln \omega_t = \rho_{\omega} \ln \omega_{t-1} + \epsilon_{\omega t},$$

(13)

where $\epsilon_{\omega t}$ is a serially uncorrelated process with zero mean and standard deviation $\sigma_{\omega}$, and $0 < \rho_{\omega} < 1$ is the first-order autoregressive coefficient.

The utility function is specified by

$$u(C_t, l_t, e_t, Q_t; z_t) = z_t \ln X_t^{\frac{1}{1-\gamma}} + \eta l_t - \frac{\phi}{2} (e_t - 1)^2,$$

(14)

where $X_t \equiv \theta C_t^{1-\gamma} + (1 - \theta) Q_t^{1-\gamma}$, and $\eta > 0$, $\phi > 0$, $0 < \theta < 1$, $\gamma > 1$.

The first term in (14) captures the notion that larger inventory holdings, $Q_t$, raise the marginal utility of any given purchase of consumption, $C_t$, either by reducing transaction costs or by better matching the consumer’s tastes. The parameter $\gamma$ is the inverse of an elasticity of substitution, which will rule the degree to which consumption and inventories are related. The shock $z_t$ will work as a preference shifter that affects the relative taste between goods (consumption and inventories) and leisure at time $t$.

The second term in (14) shows that utility is linear in leisure, as in Hansen (1985) and Rogerson (1988), which implies this utility function is consistent with any degree of intertemporal substitutability of leisure at the individual level. One of several interpretations of this property is that the average time the individual devotes to work will be constant, and all fluctuations in labor services will result from indivisible labor decisions, i.e., either full time work or no work at all. In Appendix A, I discuss in detail the implications of working with this type of preferences on leisure.

The last term in the utility function represents exactly the idea discussed above of introducing an additional effect of variable labor intensity on the agent’s preferences. The interesting feature of introducing that term this way in the utility function is that we obtain a relatively simple utility function with similar properties as those found by Burnside et al. (1993) and Burnside and Eichenbaum (1996) in their studies of labor and capital hoarding.
I assume that the preference shock behaves according to the following stochastic process

\[ \ln z_t = \rho_z \ln z_{t-1} + \epsilon_{zt}, \]  

where \( \epsilon_{zt} \) is a serially uncorrelated process with zero mean and standard deviation \( \sigma_z \), and \( 0 < \rho_z < 1 \).

I assume that total adjustment costs on investment, \( I_t h(I_t) \), are quadratic. In particular, consider \( h(I_t) = b I_t \), with \( b > 0 \).

Assume that the rate of capital depreciation is given by \( \delta(s_t) = \delta_0 + s_t^v \), with \( \delta_0 > 0 \), and \( v > 1 \). The first term represents a fixed component of the depreciation rate, while the second term refers to the endogenous part of capital depreciation, the one which depends on its use. Notice that, as \( v \) goes to infinity, the second term approaches to zero, and the endogenous depreciation rate approaches to the standard, constant rate.

In order to proceed with the calibration of the model economy, I define the length of a date to be a quarter. Hereon, all the references to the data statistics are extracted from US quarterly series, 1967:1-2008:4, seasonally adjusted and filtered (Hodrick-Prescott, \( \lambda = 1600 \)), except when indicated. The baseline values for the model’s tastes and technology parameters were chosen in the following manner.

As is standard in the business-cycle literature, the value of the discount factor \( \beta \) is set at 0.988 to match an average annual real interest rate as in the data. The capital share parameter \( \alpha \) is set to approach the average capital’s share in the U.S. national income accounts, roughly being 0.36. The steady-state ratios of both capital-to-output, \( K/Y \), and investment-to-output, \( I/Y \), are set to match the corresponding average ratios in the U.S. economy – according to the measurement of our model series, these are 7.04 and 0.23 respectively. These two values imply a depreciation rate, \( \delta \), of 0.03. In turn, for a given value of the steady-state rate of capital utilization – determined through our specific production function –, the values of both the depreciation rate and the discount factor determine a fixed component of depreciation, \( \delta_0 \), of 0.01 and an elasticity of depreciation, \( v \), of 2. The preference parameter \( \eta \) is chosen so that in equilibrium the agent spends on average approximately one-third of the time working. The value that results following this criterion is \( \eta = 2.50 \). The parameter \( \theta \), representing the weigh of the consumption good in the CES bundle of (14), is set to obtain an inventory-to-sales (I-S) ratio roughly in line with the average in the U.S. In this sense, I set \( \theta = 0.98 \), which approximately corresponds to an average I-S ratio of 0.8. Letting the steady state consumption-to-inventory ratio be as the

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8 Appendix B includes the steady-state expressions and the system’s conditions which this calibration was based on, and a summary table with all the parameter values.

9 Inventories and Sales are measured by the BEA’s series on real-private-inventories and on final sales of
average observed in the data (0.90), the previously calibrated parameters imply $\gamma = 5$ and $b = 1$.

Regarding the substitutability between consumption and inventories, the model requires $\gamma > 1$ for inventories to complement consumption. Although Kahn et al. (2002) state that inventories are procyclical for sufficiently large values of $\gamma$ (they set $\gamma = 10$), notice that the calibration of this model allows inventories to be procyclical at a lower value.$^{10}$

There is only one free parameter left to calibrate. Parameter $\phi$, related to the magnitude of adjustment costs on labor intensity, does not affect the system’s steady state but it does affect the volatilities, essentially that of the labor intensive margin. This parameter was set at 0.5 to match approximately the volatility of the average-weekly-hours series of the BLS.

The shocks’ standard deviations, $\sigma_\omega$ and $\sigma_z$, and the autoregressive coefficients, $\rho_\omega$ and $\rho_z$, are calibrated as follows. Given the model’s TFP formulation, the log sequence of technological innovations, $\omega_t$, is identified from the data by applying a growth accounting equation that differs from standard in that it introduces variable utilization of both capital and labor. Then, the process is calculated as the difference between the log of GNP and the logs of capital and labor services. Capital services are measured by using the Federal Reserve Board’s industry index of capacity utilization and the BEA’s series of real stock of capital. Labor services are measured by using the BLS’ average weekly hours and the employment series. According to this exercise, $\sigma_\omega = 0.007$ and $\rho_\omega = 0.95$ generated the volatility and the autocorrelation of the TFP process.

The log sequence of preference shocks, $z_t$, is identified from the data by using equation (8) and its particular form described in Appendix B, equation (B.2). The series is obtained, up to a constant, from the following equation:

$$\ln z_t = \ln X_t + \gamma \ln C_t - \ln \left( \frac{Y_t}{\epsilon_t N_t} \right) + \ln \left[ \frac{\phi(\epsilon_t - 1)}{N_t} \right] + \eta$$

(16)

The parameter values that generated the volatility and the autocorrelation of the preceding process were $\sigma_z = 0.01$ and $\rho_z = 0.95$.\textsuperscript{11}

domestic product. In the model, Sales are simply defined as consumption plus capital investment, which is equivalent to the difference between output and inventory investment.

\textsuperscript{10}In my doctoral dissertation (2008, ch. IV), I also contemplated the case where the utility function was separable in consumption and inventories. That setting can be approximated, indeed, as a special case where the inverse elasticity-of-substitution, $\gamma$, approaches to one. The robustness test showed that main conclusions remain valid.

\textsuperscript{11}Both the calibrated, baseline standard deviations – measured in percent deviations of innovations from the steady-state – and both the calibrated, baseline AR coefficients are consistent with the business-cycle literature – see, e.g., Arias et al. (2007). It is worth noting, though, that the benchmark model and most of the parameters will be subject to robustness and sensitivity analysis below.
4.2 Model Solution

At this point, I make use of the specific functional forms (12) and (14), the stochastic processes (13) and (15), and the parameter values introduced in the preceding section in order to find the model solution. By plugging these specifications into the economy’s resource constraint (2), the law of motion for capital (1), and the optimality conditions (7)-(11), one obtains a system of nine equations which completely describe the equilibrium behavior of the model’s variables.

In principle, it is not possible to solve this model analytically, given the strong non-linearity of the system’s equations and its relatively large dimension. Therefore, I use a pure perturbation approach as in Schmitt-Grohé and Uribe (2004) to obtain an approximate solution for the agent’s decision rules. Our model could be summarized by a collection of equilibrium conditions that take the following general form:

\[
E_t \{ f(y_{t+1}, y_t, y_{t-1}, \Gamma_t) \} = 0, \\
\Gamma_t = \rho \Gamma_{t-1} + \epsilon_t, \\
E(\epsilon_t) = 0, \\
E(\epsilon_t \epsilon_t') = \Sigma_{\epsilon},
\]

where \( y \) is the vector of endogenous variables \((C, Q, e, s, N, K)\), \( \Gamma \) is the vector of exogenous, stochastic shocks \((\omega, z)\), and \( \epsilon \) is the vector of innovations \((\sigma_z, \sigma_\omega)\).

A solution to this problem is a set of equations that relate variables in the current period with past state variables and current shocks, where the original system is satisfied. The resulting policy function can be written as

\[
y_t = g(y_{t-1}, \Gamma_t).
\]

A first-order linearization of the function \( g \), around the non-stochastic steady state \( \bar{y} \), yields

\[
y_t = \bar{y} + g_y \hat{y} + g_\Gamma \hat{\Gamma},
\]

where \( \hat{y} = y_{t-1} - \bar{y} \), \( \hat{\Gamma} = \Gamma_t - \bar{\Gamma} \), \( g_y = \frac{\partial g}{\partial y_{t-1}} \), and \( g_\Gamma = \frac{\partial g}{\partial \Gamma_t} \). For details on the properties of the function \( g \), see, e.g., Collard and Juillard (2001) or Schmitt-Grohé and Uribe (2004).\(^{12}\)

In the three sub-sections that follow, I will study the central question that motivates this work, that is the relationship between inventories and the intensity of use of the factors.

\(^{12}\)In sub-section 4.5, I discuss the properties of approximating the policy function up to a second order. For now, I will focus on the main results of this model, say impulse-response functions and first- and second-order moments, those that are not especially sensitive to the order of approximation. In particular, stochastic steady-state values and welfare comparisons are relevant subjects for this matter.
of production. For this purpose, I will first study the impulse-response functions that the model generates. In particular, these are the result of iterating the policy function (18) starting from an initial value given by the steady state. Second, I will analyze the second-order properties of the model’s variables, and compare them with the U.S. facts on inventories and capacity utilization. Third, I will discuss how uncertainty affects the results of our model, and, in particular, how important it is to consider a stochastic specification as opposed to a simpler deterministic form.\footnote{For the steady-state expressions and other details derived from the model, see Appendix B.}

4.3 Impulse-Response Analysis

In this sub-section, I describe the results obtained by studying the impulse-response functions brought by the model. The goal here is to characterize in detail the equilibrium properties of this economy. I define a \textit{shock} as the unanticipated, one-date increase of one percent in the \textit{i.i.d.} component of an exogenous, stochastic process. As the approximate solution to the model is defined, one can perform impulse-response analysis on a particular shock, suppose $\epsilon_\omega$, by assigning $\epsilon_\omega = 0.01$ at $t = 1$, while setting $\epsilon_\omega = 0$ for $t \geq 2$, and $\epsilon_z = 0$ for all $t$. The analysis that follows will show impulse-response functions for inventory holdings, the rates of labor and capital utilization, and labor and capital services, with the series expressed as percent deviations from steady state. I distinguish between high-persistence technological and preference shocks, with $\rho = 0.95$, and relatively low-persistence shocks, in particular $\rho = 0.7$.

4.3.1 The Benchmark Model

An expansionary, highly-persistent technology shock leads to an increase in both the stock of inventories and the rates of capacity utilization (see Figure 3).\footnote{By rates of capacity utilization I mean the rates of utilization of both capital and labor. Remember that capacity in this work refers to both factors of production.} Inventory holdings rise for the agent accumulates goods that are produced more efficiently during the time when productivity is higher. Notice that the rate of labor utilization, $\ell_t$, rises at the first date, but declines immediately toward its steady state value, $\overline{\ell} = 1$. What is happening is that the \textit{extensive-margin} decision taken by the agent regarding the allocation of time at $t$ was already made before the shock is known. Later on, a positive “supply” shock makes the agent increase the intensity of use of labor (the \textit{intensive margin}) even though it implies an extra cost (similar to the increased unit-cost from an extra-hour in the case of the firm). This only occurs for one date since by the next date the agent reoptimizes so that total amount of labor supplied (the \textit{extensive margin}) is adjusted without costs.
After a positive, high-persistence preference shock toward consumption and inventories, the rates of capacity utilization increase (similarly as in the technology-shock case), while the response of inventories is negative only for the very short run (see Figure 3). Now the agent prefers to consume more in the present than in the future, thus in the very short run, when the labor force cannot be adjusted extensively (just intensively, though with a cost), the inventory stock falls because part of it is just consumed. Later, when the labor force has increased, inventory holdings will also increase at higher rates because they must recover their target level - remember they are complementary to consumption.

A digression is in order at this point. Given the stochastic specification of this model, preference shocks – or loosely speaking, demand shocks - affect the marginal utilities of both consumption and inventories through the CES bundle in $u$. Then, one observes two effects here. On the one hand, since inventories act as a buffer for consumption, they could fall at impact as is shown in the figure for the baseline parameters. On the other hand, since both marginal utilities go up, both inventories and consumption could indeed increase at impact due to the optimal agent’s response. The latter may dominate the first effect under the following conditions.

Consider relatively low costs of adjustment on either the intensive margin of labor or investment (parameters $\phi$ and $b$ respectively). Let me set for a moment, and without loss of generality, $\phi = b = 0.2$, instead of the baseline parameters $\phi = 0.5$ and $b = 1$. Figure 3 also shows how the behavior of inventories and capacity utilization change. The first, and perhaps obvious, is that the rate of labor intensity goes strongly up, now that its costs are lower. The second, remarkable finding is that inventories increase after a demand shock indeed, a result that may appear counterintuitive. What is happening, though, is not at odds with the real world. Suppose an economy with either low adjustment costs on both labor contracts and investment plans, or very low adjustment costs on one of them. Now, suppose that all of a sudden individuals prefer more goods amid a decrease in the preference for leisure. In a state where labor and capital are already given, and in an economy where those adjustment costs are high, a demand shock would lead to the immediate depletion of some of the inventory holdings. That is, agents choose neither to increase production nor to deplete part of their capital because of the relatively high costs. On the other hand, an economy with lower costs will be tended to increase the intensity of use of labor, decrease part of their capital, and still raise some of their inventory holdings, as is shown in the figure.

Notice that this particular behavior of inventories is obtained without modifying the baseline inverse-elasticity of substitution between consumption and inventories (recall, $\gamma = 5$). In fact, if one raises this parameter, i.e., raises the complementarity between the two goods, the previous result not only remains but also is emphasized. In other words, for
higher values of $\gamma$, inventory holdings may largely increase at impact, as discussed earlier. However, the origin of this result is quite different. This means that the agent values more inventories as a complement to consumption, thus even a demand shock may lead to the desire for more inventories. Roughly speaking, in this case the role of complementarity from inventories is governing the results against their role as a buffer stock for consumption.

To sum up this digression, notice that through a simple way, with a general equilibrium model that would be standard except for the introduction of inventories and capacity utilization, one obtains results that are consistent with Kahn’s (1987) stockout-avoidance theory. Recall that his contribution was essentially to generate the stylized fact of procyclicality of inventories, in a partial-equilibrium model, without the need of non-standard assumptions as, for instance, the increasing-returns-to-scale assumption. Kahn (1987) generates inventories’ procyclicality by considering either a positive, serially-correlated uncertain demand, or firms having the possibility of backlogging excess demand, where they target a certain non-zero level of inventories. In fact, one also obtains this inventory-target in a general equilibrium model as the one studied here, without the need of demand shocks being highly persistent. Sensitivity exercises showed that under conditions as the ones described above – low adjustment costs on both investment and labor and, to a lesser extent, high complementarity between consumption and inventories – the effect of rising inventories at impact and their procyclical behavior are also observed at very low-persistence preference shocks – say $\rho \approx 0.2$.

After an expansionary, low-persistence impulse to the state of technology, both inventories and the rates of labor and capital utilization respond relatively stronger compared to the high-persistence shock (although the rate of use of capital is slightly stronger, see Figure 4). The explanation is simple: the agent is prone to produce more goods, both to consume and to store as inventories, when the shock is relatively more transitory. This way, she will use a much higher rate of the capacity in the short run. This stronger effect of the shock’s lower persistence does not occur with respect to the rate of capital utilization, mostly, because the agent uses more the rate of labor utilization to adjust production after very short-run impulses. Notice that the rate of labor intensity does not have persistence indeed, and the cost of changing it carries on for only one period. On the other hand, the cost of changing the intensity of use of capital carries on for a longer duration, for it affects capital depreciation, and thus the speed of capital accumulation.\footnote{In Section 5, I especially discuss the persistence property of the two rates of factor utilization. In there, I discuss the non-persistence property of labor intensity, and the relatively high-persistence property of capital utilization.}

An interesting result is also shown in Figure 4. In response to a low-persistence, expansionary shock to the marginal utility of both consumption and inventories, inventory
holdings sharply decrease at impact and stay below their steady state for several periods. A very short-run impulse to preferences leads to an abrupt rise in the demand that is not completely matched by the supply. The result is a fall in the stock of inventories with respect to its target level, even though the rates of capacity utilization still increase.

This property of inventories is what distinguishes them from the choice related to the use of factors of production. Whether inventory holdings decrease at impact depends on the nature of the shock – remember the special case where in fact they increase in response to a preference shock. And whether they stay below the steady state until convergence or simply fall during the very short run depends on persistence – consider the contrast between figures 3 and 4. On the other hand, notice that the intensity of use of capacity responds simply symmetrically to each shock, for any rate of persistence. Moreover, inventories and physical capital respond in a different way to the shocks considered above. In a standard, real business cycle model an expansionary, preference shock (either persistent or non-persistent) leads to an immediate fall in investment, with the stock of capital decreasing in the short run. Again, the question is why the stock of inventories, an alternative form of capital as interpreted here, may not fall as a response to a persistent shock to preferences? See, for instance, Figure 3. And, once again, as was mentioned above the answer lies on the role of inventories in this model, for they are not only stored as an alternative to capital, but also as a complement to consumption.

Figure 5 goes over high-persistence, impulse-response dynamics of total labor services and total capital services, that is, $e_t N_t$ and $s_t K_t$. It shows that the amount of services from labor rises immediately after every shock, whether it is a technology or a preference shock. This is due, first, to an increase in the rate of labor utilization $e_t$, and later on, to the accommodation of the labor extensive margin, $N_t$.

On the other hand, the dynamics of capital services show different patterns. For both shocks, there is an immediate jump of these services because at the beginning the stock of capital is used more intensively. A high-persistence technology shock generates that these services keep higher for a long time - in fact, more than 8 years in the simulations – because the later fall in the intensity of use of capital is compensated by a rise in investment to build-up new capital. In short, it is the replacement of the old capital what leads to a high and persistent level of capital services. On the contrary, after an expansionary shock to preferences, the level of capital services jumps too, yet it goes back more drastically to its steady state. This occurs because investment sharply falls on impact, causing the stock of physical capital to drop further together with its use.

Figure 6 shows the effects of shocks with a low persistence rate. After a technological impulse, the amounts of capital and labor services jump again on impact, but they decay strongly almost reaching their steady-state values in less than a year. In general, in extreme
cases where the shocks are not persistent, the rates of factor use will strongly increase at impact, but the levels of these factors will not rise later. The result makes sense, for the agent is optimizing by using more intensively her productive capacity today, and is not willing to pay more either for capital or employment being used tomorrow.

Less dramatic behaviors are seen with respect to the effect of a preference shock on both factor services. This is also an interesting result which is related to Figure 4. In response to a demand shock, the agent can increase the use of the capacity, and in fact the rates of factor utilization rise. Although the shock is of a low-persistence rate (what would imply dramatic changes of capacity utilization), what indeed happens is that responses are in fact weaker than those to high-persistence shocks. The main reason is, once more, the presence of inventories, for the agent has the possibility to deplete them in order to raise consumption in the very short run. This way, inventories serve as a substitute to capital and labor when a demand shock is relatively less persistent. This would suggest they are negatively correlated in high-frequency data. This is indeed the relationship I am searching along this study. In order to get to that, I employ more formal methods in the next section where, among other statistics, second moments are obtained. For now, consider how would capacity and inventories respond in models where only one at a time is present.

4.3.2 The Effects of Inventories and Capacity Utilization

Figures 7 and 8 (7 showing high-persistence preference shocks, 8 showing low-persistence preference shocks) show both the impulse-response functions from the benchmark model and those from models where either inventories or capacity utilization are absent. The primary goal here is to show that inventories are a substitute to the factors of production when a preference shock hits the economy. To show this, we should observe that the absence of variable capacity utilization pushes inventories to respond more strongly to a demand shock. On the other hand, without the possibility of accumulating inventories both rates of use of capacity should also respond stronger to this type of shocks.

Figure 7 first shows the two series of inventories in response to a high-persistence demand shock. It is clear that if capacity utilization is not allowed to vary in the model, then inventories’ response is larger. That is, they fall farther from the steady state in response to an expansionary preference shock. The same occurs with labor utilization when there are no inventories, but somehow different is the response of the intensity of use of capital. About the latter, we note that the impact effect is slightly stronger in absence of inventories. However, after the first date the presence of inventories makes the rate of use of capital rise even higher. The reason is simply found in the fact that inventories have to be replenished so that the agent follows her inventory-target. Thus, the presence of these zero-return
assets causes that capital be used even more intensively in the short run. Why does not it happen with labor utilization? Again, because labor utilization here lacks of persistence itself, which leads the agent to adjust it only for one period. And, at the first date, what happens is that the presence of inventories allows the agent to use them instead of increasing the rate of labor use, even though the stocks have to be replenish later on.

Similar results are observed when the preference shocks are less persistent, even though the differences between the benchmark model and the corresponding alternative specifications are larger. Consider Figure 8. The response of inventories to a 1% rise in the preference parameter almost reaches -0.4% when capacity utilization is fixed. But most importantly notice that the absence of variable capacity utilization raises the persistence of inventories. This, indeed, is an important result, in particular for analysis on monetary policy and the inflation-persistence issue. Furthermore, it becomes relevant for it implies that most studies on inventories that do not consider variable rates of utilization may overestimate their intrinsic rates of persistence.

The rate of labor utilization also responds stronger when inventories are absent, but the rate of capital use once again only responds more strongly at impact. Notice that the persistence of the last variable is almost the same in both cases. The conclusion is that the rate of capital use is not as substitute to inventories as it is the rate of labor utilization. But again, this is happening because the former affects a state variable as the stock of capital. And as such, it becomes more relevant in the long run when inventories indeed need to be back to their steady state level.

Consider now the effect of technology shocks where either variable capacity utilization or inventories are absent from the specification. This is shown in figures 9 (high-persistence shocks) and 10 (low-persistence shocks). As one may expect, the response of inventories is lower when variable capacity utilization is absent from the specification. This occurs when shocks are either of low or of high persistence. The reason is simple. Inventories cannot be raised as much as the agent would desire if capacity utilization is not flexible in the very short run. In fact, in response to an expansionary, high-persistence technology shock inventories fall in the short run. One must have in mind that a persistent increase in productivity raises the opportunity cost of having inventories, which in the absence of a flexible rate of use of the factors of production leads to their transitory depletion. The same happens with both rates of utilization.

### 4.4 The Association Among Variables

In this sub-section, I briefly show the U.S. evidence on capacity utilization and inventories, and study both the relationship among the variables in the model and their volatility. These
concepts are summarized by the correlation between any pairs of variables, and their relative standard deviations. Briefly, these are derived as follows. First, we take expectations to the policy function described in its general form by \( y_t = \bar{y} + g_y \hat{y} + g_T \hat{T} \), to obtain \( E\{y_t\} = \bar{y} \). Then, we take the second moment of the vector of endogenous variables \( \{y_t\} \), in order to obtain the covariance matrix \( \Sigma_y = E\{y_t - \bar{y}\}\{y_t - \bar{y}\}' \). This matrix is found through an expression of the form \( \Sigma_y = g_y \Sigma_y g_y' + g_T \Sigma_T g_T' \). The covariance matrix is used, finally, to find the correlations among the model’s variables and the relative standard deviations.

4.4.1 The Benchmark Model

Table 1 summarizes the cyclical behavior of inventory holdings, the inventory-to-sales ratio, and the services from capital and labor.\(^{17}\) First, note that the correlation between inventories and output is positive (0.57), while that between the inventory-to-sales ratio and output is negative (-0.39). Second, notice that the correlations of inventories with both capital services and labor services are positive (0.56 and 0.66, respectively). Finally, while the volatilities of inventories, the inventory-to-sales ratio, and labor services are lower than that of output, the relative standard deviation of capital services doubles it. We will observe below that the model fails in replicating this fact, for that it cannot account for such a high variance of the services from capital.

Tables 2-3 show the results based on the covariance matrix of the model’s approximated policy function. This computation included the filtering of the series by applying HP, \( \lambda = 1600 \).

Table 2 shows percent standard deviations relative to output volatility for (i) the benchmark model, (ii) a model with variable capacity utilization but no inventories, and (iii) a model with inventories but no variable capacity – where I also distinguished between the absences of variable utilization of both labor and capital. The purpose, here, is to study how the variables behave under different specifications.

Except for capital services and capital utilization, the benchmark model behaves roughly as observed in the U.S. business-cycle data. The variability of labor services, inventory holdings, and consumption are lower than the variability of output, whereas that of investment in physical capital is larger. Consistent with the evidence as well, the stock of capital has lower volatility than the series aforementioned.\(^ {18}\)

Consider for a moment what we should expect from the series’ variability in each of

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\(^{16}\)Once again, consider for now a first-order approximation to the policy function.

\(^{17}\)Remember this is U.S.’ quarterly data, during the period 1967:1-2008:4

\(^{18}\)Though not shown in Table 1, the cyclical variability of consumption is almost two third that of output, the stock of capital is approximately one fifth as variable as output, and investment is approximately three times higher.
the specifications considered in Table 2. Remember that inventories fulfil two roles in this model: they act as a buffer stock for consumption, and at the same time provide utility services. Suppose we take away the possibility that the representative agent holds inventories, and consider, first, the consequences of a demand shock. Recall that, at impact, a demand shock would lead the agent to lower inventories in order to raise consumption. However, they will rise soon to reach their target level for they complement consumption. If there are no inventories in the economy, only investment would act to buffer consumption.

Consider a positive technology shock. If there are no inventories, the rise in productivity and output would go only to consumption, instead of going to both consumption and inventories. As a consequence, the presence of inventories tends to increase the variability of output, and to smooth consumption and investment. This is what we basically observe in Table 2 by comparing the benchmark model with the no-inventories model.

Suppose now that capacity utilization is fixed. This is an economy where agents can store goods as inventories, but cannot change the services from labor and capital, once the state of nature is revealed. Note from Table 2 the different volatilities of investment under the two specifications. As one would expect, variable capacity utilization leads to a rise in the volatility of investment. The contribution of technology shocks to the variability of investment in the benchmark model is above 95%. Remember, though, that technology shocks also lead to higher use of capacity. Then, a shock of this kind generates more investment variability when there is a chance to vary the use of capacity. In fact, not only investment but most variables’ standard deviations increase in the benchmark model. Furthermore, notice that variable capital utilization leads to changes in the capital depreciation rate, which in turn generates more variation on investment, thus providing another source of volatility.

Finally, if we look at the last two columns of Table 2 we see that in fact the rate of capital utilization has the leading role in terms of raising the volatilities of the model. There is an interesting result, though, with respect to the rate of labor utilization. Note that for both the inventory-to-sales ratio (I-S) and inventory holdings the absence of variable labor use leads to a rise in their volatilities. Recall that the rate of labor utilization has no persistence in the model. Moreover, any expansionary shock causes a short-run increase in this rate, and generally a fall in inventories (a weak rise at most, depending on the nature of the shock). I will show below that these two variables will be negatively related. What we already know from the previous section is that they act as substitutes in the model. This, in general, implies that the absence of one of them causes the other’s volatility to increase.

Table 3 shows the correlation matrix from the benchmark model. The first noteworthy result is that, consistent with the evidence, output is negatively correlated with the I-
S ratio (-0.72), while positively correlated with inventory investment (0.63). As it was mentioned in the introduction, the countercyclicality of the I-S ratio and the procyclicality of inventory investment have been largely documented in the empirical inventory literature. The shortcoming here is that the countercyclicality of the I-S ratio is higher than in the evidence, which documents a correlation of -0.39.

As was mentioned above, inventories are negatively correlated with the intensity of use of labor. Notice, though, that the former are positively correlated with labor services. Once again, the lack of persistence of the intensive margin, added to the procyclicality of the extensive margin, is generating this result. On the other hand, both the rate of capital utilization and total capital services are positively correlated with inventories. Finally, the correlations arising in this model among capital, labor, consumption, and investment do not differ much with respect to those arising in standard business-cycle models.

4.4.2 The Relationship Between Capacity Utilization and Inventories, and the Source of Shocks

I study now the association between inventories and capacity utilization by source of uncertainty and degree of persistence. Table 4 shows the correlations among the three main variables distinguished by the two shocks and two extreme rates of persistence: $\rho = 0$ and $\rho = 0.95$.

As we observe from the table, low-persistence preference shocks make the correlation between inventories and the two rates of capacity utilization be negative. They simply act as substitute goods, which is something I already mentioned above. That is, non-persistent demand shocks lead inventories and utilization respond in an opposite way. But notice that as the shock’s persistence goes up, this association becomes positive for the rate of capital utilization. Once again, as the duration of the preference shock is higher, capacity moves together with inventories as the latter are not allowed to fall farther from their target. Notice that this does not happen with the rate of labor utilization – its correlation with inventories is -0.62. Once again, it is the implication that, in response to an expansionary shock for instance, the agent raises this rate only for one date, which indeed is the date when inventories fall whatever the shock’s persistence. At the end, they will be negatively correlated at every persistence. However, given the relatively higher persistence of the services from labor, they will be positively correlated with inventories under demand uncertainty, as will also be capital services.

Notice that the correlation between the two capacity variables is positive when uncertainty is governed by non-persistent preference shocks. Yet, it becomes almost negligible under the presence of highly persistent preference shocks. Once more, the lack of persis-
tence of the rate of labor utilization leads through this result, for the services from these two variables are still positively correlated – though not shown in Table 4.

When uncertainty is led by technology shocks only, the correlation between inventories and capacity is clearly positive if these are non-persistent. Explanations are the same as those given above. Recall that an expansionary technology shock makes these two concepts complement each other. That is, when productivity is higher the agent takes advantage of it and produces more of the three goods available in this economy: consumption, inventories and capital. When these shocks are more persistent, though, only the rate of capital utilization keeps its positive correlation with inventories. About the correlation between the two rates of utilization, their positive and strong association when shocks are non-persistent contrasts with their weaker positive correlation when shocks are highly-persistent. Simply, the lack of persistence of labor utilization explains this result.

Table 5 shows the relationship between inventories and the rates of capacity utilization from alternative specifications. The purpose of this exercise is to isolate two variables at a time in order to have their association when the third variable of interest is not in the the agent’s choice set. For example, consider the correlation between inventories and the rate of capital use when preference shocks are non-persistent: this is of -0.60. It shows that in absence of variable labor utilization, inventories and capital utilization are negatively associated when uncertainty comes from these shocks only. That rate was in fact -0.84 if labor was allowed to vary. This means that the substitution between inventories and the variable rate of capital use is lower when labor utilization becomes fixed. The result basically shows that the two rates of capacity are in fact highly complementary to each other, and, when both available, they make inventories raise their substitution role in response to demand shocks. This also happens in regards with the correlation between inventories and the rate of labor utilization. A remarkable result emerges when one observes the correlation between the two rates of capacity utilization when inventories are not among the agent’s choices. When demand shocks are non-persistent, this association almost reaches the 100% level, whereas in the benchmark model it was only 0.47. This pattern is also present under high-persistence preference shocks. Once again, this confirms the idea of inventories being a substitute for capacity utilization. When they are introduced into the model, they help lower the response of capacity utilization when a preference shock hits the economy. Regarding technology shocks, we observe that the absence of a third variable does not affect much the results. This is not surprising, since the effect of technology shocks is much clear and simpler than that of preference shocks.
4.5 Does Uncertainty Matter?

As we studied in the preceding sections, the solution to the benchmark model is found by applying a first-order approximation to the system of necessary conditions. Even though in general, and for this model in particular, behavioral results are not especially sensitive to higher-order approximations, welfare comparisons can indeed be misleading this way – see, for example, Kim and Kim (2003) and Schmitt-Grohé and Uribe (2004). An undesirable feature of the standard log-linearization is that it does not break the certainty equivalence property. The problem is that second- and higher-order terms of the equilibrium system are omitted, and thus it becomes impossible to obtain the effect of volatility in the model. On the contrary, a second-order approximation to the policy function allows for second-order terms to arise, causing volatility to affect the model’s solution.

A corollary of this statement is that by linearizing our stochastic, benchmark specification up to a first order, in essence it becomes deterministic. For instance, the steady state found in order to linearize the system of necessary conditions was indeed a non-stochastic steady state, and the resulting impulse-response functions were basically algebraic forward iterations of the model’s decision rules.

When the policy function is approximated up to a second order through the application, for instance, of the perturbation method developed by Schmitt-Grohé and Uribe (2004), one can have a better description of what is happening in our model in terms of uncertainty. In principle, it will serve to find a stochastic steady state for the economy, i.e., the state to which all variables converge when one accounts for the effect of uncertainty. In general, as the variance of the exogenous shocks rise, the stochastic-steady state values get farther from the non-stochastic steady state. In particular, a second order-approximation will help us find the effect of increasing uncertainty on our variables’ stochastic steady states, and especially on inventories and capacity utilization.

By linearizing up to a second order, impulse response functions will become the result of performing Monte Carlo simulations on our benchmark model. The technique implies pulling shocks from their distributions and evaluate how they affect the system, repeating this experiment a large number of times so as to sketch an average response. On the other hand, stochastic steady-state values are the result of taking expectations to the second-order-approximated decision rules, which in turn bring about the non-stochastic steady states corrected by the effects of second-order moments, i.e., the auto-covariances – see, e.g., Schmitt-Grohé and Uribe (2004).\(^{19}\)

Figure 11 shows the stochastic steady-state values arising from different combinations

\(^{19}\)These exercises were performed by employing Dynare v3, a set of codes used to solve and estimate DSGE models.
We notice that the steady-state inventory-to-output ratio rises with higher volatility either from preference or technology shocks. We should expect this to happen since one of the motives to holding inventories comes from uncertainty, i.e., recall the stock-out avoidance motive leads the agent to store goods in order to protect herself from different realizations of the shocks. Thus, the stochastic steady-state inventory-to-output ratio is about 84% in an economy with no uncertainty, while it exceeds 100% when both shocks have standard deviations of 10%. Notice that technology shocks have a relatively higher impact on this ratio, for a technology volatility of 10% causes this ratio to increase to about 98% if there is no any volatility on preferences. On the other hand, the ratio does not exceed 90% when the standard deviation of preference shocks reaches 10% and there is no variance on technology shocks. This confirms the idea that inventories are partly held by a stock-out avoidance motive: an increase in the volatility of the preference shock – a demand shock, essentially – causes the agent to wish to hold a higher stock of inventories, in average, to protect herself against expected larger demand shocks.

The stochastic steady-state inventory-to-consumption ratio behaves similarly, indeed, as the inventory-to-output ratio. It is worth noting that, although not shown, the stochastic steady-state consumption-to-output ratio declines with increases in volatility. Yet, these reductions are weaker compared to the changes on the inventory ratios (they slightly fall from 75.6% when no any volatility to 74.6% when both are 10%). In this case, if one individualizes the changes on the standard deviations, once again technology shocks are clearly more important. This makes sense in that a larger size of technology shocks – supply shocks, essentially – leads the agent to allocate output and time not only onto higher amounts of capital and labor but also to improve the inventory stock. The agent may be engaging in precautionary saving that affects average consumption. For instance, think on an expected, large negative supply shock. The effect on the agent’s decisions is simple: she will surely choose to hold a higher amount of inventories, once more to shield herself later. It is the potentially high cost of not having enough resources to consume after a bad realization what leads to these results. At the end, the average stocks of both capital and inventories would rise with respect to the average flows of output and consumption when uncertainty goes up.

The stochastic steady-state rates of both capital and labor utilization slightly fall as volatility goes up. The former falls from 14.88% when no volatility to 14.21% when both reach 10%. The latter falls even less, from 1.000 without variance to 0.997 if standard deviations are 10%. Although small, explanations for these falls could be found in the increases of the average levels of capital and labor. A higher level of uncertainty leads to increases in average precautionary stocks leaving their average rates of utilization either
unchanged or slightly lower. Though not shown here, the average gross investment-to-output ratio also falls slightly with the increase in uncertainty, in spite of the average stock of capital being higher. This occurs because the average depreciation rate falls when volatility grows, which simply comes from utilizing capital at lower intensity.

Thus, a conclusion one might be tempted to assert is the following. For otherwise identical economies, those with high variance are most likely to have lower averages in the consumption-to-output and the investment-to-output ratios, while higher averages in their inventory-to-output and capital-to-output ratios.

As I mentioned above, a second-order approximation to the equilibrium system allows us to capture the effects of uncertainty on welfare. I now proceed to derive an approximation to the welfare function, one which becomes related, among other things, to uncertainty. To this matter, I will follow Collard and Dellas’ (2006) procedure to comparing welfare results.

First, the welfare function is computed taking a second-order approximation to the utility function. Second, I compute first and second order theoretical moments for its arguments: consumption, leisure, inventories, and the rate of labor intensity. There is a difference here with respect to Collard and Dellas (2006), since they compute these moments from a multitude of many-period simulations of the model’s series. Results of course should not be different the other way, simply because the theoretical moments are the average values to which the simulation-based results are supposed to converge. Finally, these moments are then entered into the approximated utility function so as to compute the unconditional expected welfare.

A second-order Taylor approximation to the utility function (14) brought about the following result:

\[
\begin{align*}
    u & \approx \bar{u} + z_0(z_t - \bar{z}) + c_0(C_t - \bar{C}) + q_0(Q_t - \bar{Q}) + n_0(N_t - \bar{N}) \\
    & + e_0(e_t - \bar{e}) + \frac{1}{2} \left[ c_1(C_t - \bar{C})^2 + q_1(Q_t - \bar{Q})^2 + e_1(e_t - \bar{e})^2 + c_0 \right],
\end{align*}
\]

where,

\[
\begin{align*}
    z_0 & = \ln \bar{X}^{1/(1-\gamma)}, \quad c_0 = \left( \frac{\bar{z}}{\bar{X}} \right) \theta \bar{C}^{-\gamma}, \quad q_0 = \left( \frac{\bar{z}}{\bar{X}} \right) (1 - \theta) \bar{Q}^{-\gamma}, \quad n_0 = -\eta \bar{N}, \\
    e_0 & = -\eta \bar{N} - \phi(\bar{e} - 1), \quad c_1 = (-\bar{C}^{-\gamma} \bar{X})^{-2} \left[ \gamma \theta (1 - \theta) (\bar{Q}/\bar{C})^{1-\gamma} + \theta^2 \right], \\
    q_1 & = (-\bar{Q}^{-\gamma} \bar{X})^{-2} \left[ \gamma \theta (1 - \theta) (\bar{C}/\bar{Q})^{1-\gamma} + \theta^2 \right], \quad e_1 = -\phi, \\
    c_0 & = \theta \bar{C}^{-\gamma} \bar{X}^{-1}(z_t - \bar{z})(C_t - \bar{C}) + 2(1 - \theta) \bar{Q}^{-\gamma} \bar{X}^{-1}(z_t - \bar{z})(Q_t - \bar{Q}) \\
    & - 2\eta(e_t - \bar{e})(N_t - \bar{N}) - 2\bar{z} \theta \bar{C}^{-\gamma} \bar{X}^{-2}(1 - \theta) \bar{Q}^{-\gamma}(1 - \gamma)(C_t - \bar{C})(Q_t - \bar{Q}),
\end{align*}
\]

and an over-bar indicates that the variable or function is evaluated at the stochastic steady
state.

Taking the expectation to the approximated expression for the utility function in the right hand side (RHS) of (19) – denoted by \( u^* \) herein – one obtains the following:

\[
E(u^*) = \bar{u} + \frac{1}{2} \left[ c_1 \sigma_e^2 + q_1 \sigma_q^2 + e_1 \sigma_e^2 + 2 \text{cov} \right],
\]

(20)

where I use the facts that \( E(x_t - \bar{x}) = 0 \), and \( E(x_t - \bar{x})^2 = \sigma_x^2 \); while \( E(x_t - \bar{x})(y_t - \bar{y}) = \sigma_{xy} \) is used inside the term \( \text{cov} \).

Notice that coefficients \( c_1, q_1, \) and \( e_1 \) are all negative, yet we do not know the sign of the whole term comprised by covariances. For special conditions in which the last term becomes negligible, our result would imply that the higher the variables’ auto-covariances, the lower the (approximated) unconditional expected welfare. A question emerges here, though, because the value of utility at the stochastic steady state, \( \bar{u} \), changes with the shocks’ standard deviations. In general, higher variances of the shocks imply higher variances of the endogenous variables. At the same time, they imply different averages for these variables, i.e., stochastic steady states, which in turn affect \( \bar{u} \). Therefore, in principle we cannot say that increasing the model’s exogenous variance implies lowering welfare. Indeed, from the simulations to the benchmark model we notice that increases only in the volatility of preference shocks lower welfare, while rises in the volatility of technology shocks go the other way around. That is, an increase in the variance of the technology shocks produces an increase in the expected welfare of the agent.

Finally, notice that were not for the second-order linearization of the utility function, we would only had \( E(u^*) = \bar{u} \). Assuming the latter as true, and taking welfare comparisons based on it, would lead us to misinterpret the stochastic setting as a deterministic one. There is a vast recent literature – see, e.g., Kim and Kim (2003) and the references therein – discussing another inconsistency, which is that of comparing welfare by approximating the utility function up to a second order while approximating the policy function to a first order.

The lesson one can take from the previous discussion is that second-order approximations enhance our understanding of the behavior of capacity utilization, and most importantly of inventories. In fact, the latter are essentially closely related to the concept of uncertainty. We have seen that higher uncertainty implies, ceteris paribus, a higher inventory stock, but lower rates of capacity utilization, in average. The question of whether higher uncertainty would lead to lower welfare, though, is still open.

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5 Robustness Analysis: A Comparison with an Alternative Specification On Capital Utilization

This section discusses the robustness of the results by comparing the way the benchmark model treats variable capital utilization with an alternative setting introduced by Christiano et al. (2005). Our benchmark economic environment was built on the basis that variable capital utilization goes together with the assumption of endogenous depreciation. That is, the stock of capital is used to produce goods at different rates of intensity according to optimality conditions, and this intensity of use affects the depreciation rate. Consider equation (9) in Section 3. For a given level of capital, raising the flow of capital services by one unit basically implies the following trade-off. There is a benefit arising from the marginal product of capital services (LHS), while there is a cost in higher depreciation of the current stock of capital, valued in terms of consumption goods (RHS).

Although most of the literature on variable capacity utilization generally assumes the specification of endogenizing a capital depreciation rate, there are alternative specifications to treating the cost of varying the use of capital. A recent standard example is the set up by Christiano et al. (2005) (CEE specification herein), mentioned briefly in the literature review. In that paper, variable capital utilization plays an important role in explaining the observed inertia in inflation and persistence in output. However, their theoretical model assumes a constant depreciation rate, independent of the rate of capital utilization. Instead, they model the extra-cost of using capital more intensively with respect to the steady-state rate by an increasing, convex function of $s_t$, denoted herein by $a(s_t)$.

Thus, for a given stock of capital, varying capital services by one unit does not raise the depreciation rate, but involves a cost to the agent in terms of resources of $a'(s_t)K_t$. In CEE, the functional form for $a(s_t)$ is set to require that the rate of capital use, $s_t$, be 1 in steady state. Further, they assume that $a(1) = 0$. These two assumptions imply, first, that the steady-state stock of capital will equal the flow of capital services and, second, that the additional costs that depend on the rate of capital utilization, $a(s_t)K_t$, disappear from the resource constraint in steady state. Therefore, they are somehow modeling capital utilization in a similar fashion as labor utilization is modeled here. That is, recall that the labor extensive margin is assumed to be decided before the state of nature is known, while the labor intensive margin is set after the state of nature is realized. Also, remember that the costs of increasing both margins are the same in terms of disutility, except for an additional cost that only involves the intensive margin. This extra-cost in terms of disutility was assumed to have the form $\frac{\sigma}{2}(e_t - 1)^2$, which implies a steady-state rate $e = 1$, and extra-costs - not in terms of resources as in CEE but in terms of welfare – arising from deviations when adjusting labor decisions ex-post.
The similarity between CEE’s way of modeling capital utilization and this work’s way of modeling labor utilization is evident. There is an interesting difference in their results that warrants a brief discussion though. In Section 3, we noticed that the way labor utilization is modeled leads to responses in the rate of labor intensity that are not persistent, even for highly persistent shocks. And remember, this result is simple to understand. Because of the additional adjustment cost on the intensive margin, any persistent shock – either to preferences or technology – makes the agent change only temporarily the rate of labor utilization, while labor is given by pre-shock decisions. In absence of further shocks, the agent will only adjust the extensive margin by the second period, simply because it is cheaper in terms of disutility.

One might presume that this effect will also be present in CEE with respect to the rate of capital utilization. That is, suppose any shock, of any rate of persistence. At first, the agent would immediately respond by optimally adjusting the rate of capital utilization, given a certain level of capital. In absence of further shocks, the agent will take the decision between keeping the capital utilization rate outside the steady state and adjusting the stock of capital. The first involves an additional cost in terms of resources, given by $a(s_t)K_t$, while the second implies postponing consumption. Similar to the effect on labor utilization observed in Section 4, one may think that the agent will choose to change only the stock of capital, depending on the adjustment costs to investment in some way. But the conclusion is that, even for null adjustment costs to investment and zero-persistence shocks, in most situations the agent will still choose a different rate of capital utilization from that of the steady state, even for several periods after the shock. In other words, as opposed to the effect of no-persistence on the rate of labor utilization studied so far, the rate of capital utilization in a CEE specification does have persistence indeed.

Why does the latter take place? To answer this question, one has to look at the nature of the investment decision, which is at the heart of the recursiveness of the problem. This is quite different from the labor decision, which is essentially an intra-temporal decision. As I show below, specifying variable capital utilization à la CEE in this work does also generate a rate of capital utilization that is persistent, even in responses to shocks that are not persistent at all. In fact, either by looking at the first-order autocorrelation coefficient, or by looking at the behavior of the impulse-response function (e.g., finding the half-life of the series, defined as the number of periods it remains above 0.5% after a unit-shock), one cannot observe any differences between CEE and benchmark regarding the rate of capital utilization – see Table 6 and Figure 12. If there is any difference at all, notice from Table 6 that CEE displays higher autocorrelation coefficients, which provides some idea that it is quite persistent – though I do not test any of these hypotheses here.

An alternative specification to the benchmark model in line with CEE would imply the
following changes. The law of motion for capital, equation (1), would now be

\[ K_{t+1} = K_t (1 - \delta) + I_t, \]  

(21)

where \( \delta \) is the constant capital depreciation rate. The resource constraint (2) would now be

\[ C_t + I_t [1 + h(I_t)] + Q_{t+1} - Q_t + a(s_t)K_t \leq Y_t, \]  

(22)

where \( a(s_t) \) is defined as mentioned above.

In the new specification, the first order-necessary condition (9) changes to the following equation

\[ F_2(e_tN_t, s_tK_t) = a'(s_t), \]  

(23)

whereas (10) becomes

\[ u_1(C_t, l_t, e_t, Q_t) p_{k,t} = \beta E \left\{ \frac{u_1(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1}) \times s_{t+1}F_2(e_{t+1}N_{t+1}, s_{t+1}K_{t+1}) + (1 - \delta)p_{k,t+1} - a(s_{t+1})}{|\Gamma_t|} \right\}. \]  

(24)

Notice that equation (23) only differs from (9) on the right hand side. Now the decision includes the marginal cost, in units of consumption, of increasing the intensity of use of capital: \( a'(s_t) \). In a different way, the decision before included the marginal cost in terms of capital deterioration, \( \delta'(s_t) \), transformed to units of consumption good by \( p_{k,t} \). On the other hand, equation (24) introduces now the term \( a(s_{t+1}) \) on the RHS. It shows that the discounted, expected future marginal benefit of increasing physical capital is affected by an additional cost that depends on its future intensity of use. Of course, if the stock of capital is being used at its steady state, then no extra-costs will be accounted.

In order to compare the results from the benchmark model with those from the CEE model, let us assume that \( a(s_t) = 0.5v(s^2_t - 1) \). This functional form satisfies the requirements stated above, while \( v \) is a positive parameter that serves to normalize \( s_t \) to be 1 in steady state, as in CEE. Even though the benchmark setting does not normalize \( s_t \) this way, I scale CEE steady-state values so that we depart from the same position in order to compare them consistently.\(^{20}\)

Impulse-response analysis performed on the alternative, CEE specification shows that in general it does not differ from the benchmark specification. The only slight difference appears from the investment response to high-persistence preference shocks, say \( \rho_z > 0.9 \). Consider an expansionary shock. Then, the CEE’s investment impulse-response function will be at all points below the one from benchmark see Figure 12. The difference comes

\(^{20}\)See the notes about the dimensions of the rates of labor and capital utilization in Section 3.
from the fact that in the benchmark model the cost of increasing $s_t$ is “paid” with higher capital depreciation, i.e., with capital goods, whereas in the CEE-version it is paid with consumption goods taken from the resource constraint. When the cost is charged on capital depreciation, there is a need to replenish the stock of capital, thus investment does not decrease but in fact for highly-enough persistent, demand shocks it could rise. On the other hand, when the cost is paid over the agent’s resources she will be relatively more willing to lower investment in order to smooth consumption.

What happens in the end is reflected by equations (9) and (23). In both, the LHS could be interpreted as the demand for capital services for a given level of $K_t$, while the RHS could be interpreted as the supply of capital services. The latter, indeed, is a function only of $s_t$ in (23), but a function both of the marginal cost of capital depreciation and the price of capital in (9). Notice that, for given $N_t$ and $K_t$, an expansionary preference shock leads to a rightward shift of the demand for capital services arising from an increase in the intensity of use of labor, $e_t$. This, in turn, causes a movement along the supply curve that finally leads to an increase in the rate of capital utilization.

This mechanism is at work in both specifications. Yet, the difference relies on the fact that the benchmark’s supply curve includes a factor comprised by the price of capital, which solely depends on investment. Remember that in general investment falls in response to a preference shock, for the agent is prone to consume more rather than saving. It turns out, then, that the effect of a preference shock on the intensity of use of capital is muffled by a fall in investment, and vice versa; that is, the effect of a preference shock on investment is muffled by an increase in the intensity of use of capital. In addition, one shall notice that the higher the parameter $v$ is in the benchmark specification (i.e., the closer we approach to the standard, fixed-capital-utilization model) the lower the response of $s_t$ to a preference shock, since the capital-services supply schedule becomes steeper. Yet, at the same time, the magnitude of the response (decrease) of investment to a preference shock increases.

Consider now Table 7, which shows the volatilities in the CEE specification. Notice that there are no significant differences with respect to the benchmark setting. The percent volatility of investment relative to that of output is slightly lower in CEE (1.33 versus 1.73), while that related to the rate of capital utilization is slightly higher (0.51 versus 0.41). This shows a small improvement of the CEE with respect to the benchmark model, since, remember, we need that capital services be more volatile. As mentioned above, movements in investment help muffling movements in the rate of use of capital in the benchmark setting. As a result, investment is more volatile under that setting, contrarily to what happens with the rate of use of capital. Although not shown in Table 7, the standard deviation of output becomes larger in CEE as well. Finally, Table 8 shows the correlations arising from the CEE specification. Essentially, they are roughly similar to those from the benchmark model.
see also Table 3. As one would expect from earlier comments, the main differences arise from investment.

6 Conclusion

The primary goal of this work is to gain a better understanding, in the context of a general equilibrium Walrasian stochastic framework, of the role of inventories and capacity utilization (of both capital and labor) and, in particular, the relationship among them. These are variables which have long been recognized as playing an important role, and having rather well defined associations both among each other and with other indicators in the business cycle. An analysis of the association between inventories and capacity utilization seems natural, since in some conceptual sense physical capital can be seen as a stock ultimately destined to be transformed into an inventory of finished goods. Likewise, inventories could be seen as a stock of physical capital already transformed into finished goods. Furthermore, once we introduce the possibility of variable rates of utilization of the capital stock, then both such rate of utilization and inventories can be seen as providing a short-run adjustment “buffer stock” mechanism. A variable rate of utilization of labor is also introduced, in a manner that is very much symmetric to the rate of capital utilization – the symmetry not being perfect, since changes in the rate of capital utilization change a state variable (the capital stock) while the same is not true of changes in the rate of labor utilization.

In Section 3, we discussed how our model differs from those in the existing literature. One methodological difference, which does not change the ultimate results, is that since firms are competitive, then they can be abstracted from, so that the analysis can be conducted in terms of the behavior of “agents” or “households”, performing all activities (production, investment and consumption). This is also explained in detail in Section 3. Being this the case, one should see some of the assumptions as “mimicking” what we would observe for firms. One example is the assumption of utility depending not only on leisure but also on “last minute” changes in labor (leisure): for the firms, this is the usual case of changes in the number of work hours. Another is the modeling of changes in preferences, which corresponds to firms’ changes in demand. Finally, the existence of inventories in the steady state is motivated by their presence in the utility function –mimicking, to some extent, the association between the levels of final goods inventories and sales at the level of the firm.

Introducing both variable capital and labor utilization, and inventories in a dynamic, stochastic general equilibrium model generates some limitations if one is interested in obtaining an analytical solution. The relatively large number of variables and equilibrium
conditions, in addition to the presence of non-linear expressions, allows only to find approximate solutions. The analysis of the relationship between inventories and capacity utilization is centered on the effects of two possible shocks: preference (demand) shocks and technology shocks. The first is specified via shocks in a parameter of the utility function; the second, via shocks in a parameter of the production function. This allows not only to investigate the nature of those relationships, but also a discussion of the extent to which the results conform to the associations observed in the data, or the “stylized facts”.

In these impulse-response experiments, a “shock” (it is more formally defined in Section 4), means an increase, for only one date, of 1% of the random term. In each case, it will be useful to distinguish between “high” and “low” persistence of the shock. The results, on which we elaborate in more detail in Section 4, can be summarized as follows.

Consider, first, the effects of a positive technology shock. For a relatively “high” persistence of the shock, inventories behave as being complement to capital utilization (i.e., they are positively associated), but as substitute to the rate of labor utilization – the latter resulting from the lack of persistence of the rate of labor utilization. Still, inventories are positively associated with (i.e., behave as complement to) labor services. For relatively “low” persistence of the shock, inventories act as a complement to (i.e., are positively associated with) both capital and labor utilization rates.

Second, consider the effects of a preference shock – more specifically, a positive shock increasing the utility of both consumption and inventories. Here, a high-persistence shock, as in the case of a productivity shock, shows inventories as being a complement to capital utilization, but a substitute to the rate of labor utilization. Still, and as in the case of the technology shock, inventories are associated positively with (i.e., as a complement to) labor services. For shocks of relatively low persistence, inventories behave as a substitute for both rates of capacity utilization (capital and labor).

These findings warrant some further considerations. Notice, first, that inventories and the rate of capital utilization are complements in three of the four cases described above, while inventories and the rate of utilization of labor are substitutes in three of them. It is worth remembering, though, that both rates of utilization complement each other in response to any shocks of any persistence. One can infer that both rates of utilization associate differently with inventories because, as “shock absorbers”, both inventories and the rate of labor utilization result in less of an effect carried on to subsequent periods than changes in the rate of capital utilization and the depreciation rate. In other words, differences in the internal persistence of inventories and the rates of capacity utilization are

---

21 Remember that the stochastic structure of shocks is of the form $\ln s_t = \rho \ln s_{t-1} + \epsilon_t$, where $\epsilon$ is the random component and $\rho$ is the measurement of the shock’s persistence. We should also keep in mind the important difference between the persistence of a shock (measured by the coefficient $\rho$), and whatever is the “intrinsic”, internal persistence of the model.
central for understanding these results.

Second, and related to the first comment, notice that low-persistence shocks generate more clear results, in the sense that inventories complement both rates of utilization in response to technology shocks (as one would expect), while they substitute both rates in response to preference shocks (as one would also expect). High-persistence shocks, on the other hand, emphasize mostly the role of inventories as being a complement to consumption instead of emphasizing their role as a “shock absorber”.

By considering the effects of the two shocks simultaneously, where both have the baseline, calibrated persistence rates and standard deviations, one finds that inventories are associated positively with capital utilization, but negatively with labor utilization. Furthermore, inventory holdings are procyclical, while the inventory-to-sales ratio is countercyclical, being both findings consistent with the stylized facts in the literature.

Regarding the robustness of the benchmark set up, I discussed an alternative way of modeling variable capital utilization proposed by Christiano et al. (2005). We observed that, by taking into account any of these alternatives, the qualitative features remain unaltered, though quantitative results are affected.

Related to the main line of the work, an additional “theme” has been discussed in Section 4. It has to do with the treatment of uncertainty and the consequences of using, as it is done in most of the literature, a first-order approximation to the decision rules. As indicated before, the complexity of the model makes it impossible to derive a closed solution. Being the model stochastic, it also implies that some features are affected by the way in which the solution is approximated. In general, a first-order approximation to the model brings satisfactory results if one is interested in characterizing the variables’ dynamics. However, second- and higher-order approximations do become relevant when studying stochastic steady-state values and making welfare comparisons. In particular, by applying a second-order approximation, we can analyze more accurately the behavior of the stochastic steady-state values of both inventories and capacity utilization. Interestingly, as uncertainty rises, the average of inventories increases relative to consumption and output, whereas the averages of both rates of utilization fall. As one would expect, higher exogenous uncertainty enhances the importance of the supposedly most relevant motive to holding inventories: the precautionary motive. Welfare comparisons were also performed by approximating to a second order both the decision rules and the utility function. Results here are mixed. As the volatility of the preference shock rises, expected welfare falls. As the volatility of the technology shock rises, then expected welfare would increase. This issue remains to be explored more carefully. Nonetheless, it is left for further research.
References


APPENDIX

A Firms and the Competitive Equilibrium

There are a number of market specifications that can support the allocation arising from the representative-agent problem. In order to keep things simple, assume that the typical household maximizes her lifetime utility by choosing contingency plans for consumption, inventory holdings, capital investment (and with it the rate of capital utilization), and the supply of labor services. On the other hand, the firm maximizes profits by choosing at every period an expected demand for capital services, \( S_t^d \) – in equilibrium, equal to \( s_t K_t \) –, and an expected demand for labor services, \( L_t^d \) – in equilibrium, equal to \( e_t N_t \).

I will now describe a competitive-equilibrium specification consistent with Hansen’s (1985) and Rogerson’s (1988) indivisible-labor models.\(^{22}\) I assume that, before the state of nature is known, there is a market for state-contingent labor services where households choose their expected supply of labor – the extensive margin – and an associated state-contingent rate of labor utilization – the intensive margin. The difference with Hansen-Rogerson model is that, in this economy, at the beginning of every period \( t \) firms and labor suppliers face a state-contingent wage schedule that specifies the wage as a function of the information set \( \Gamma_t \equiv (\omega_t, z_t) \). This means that the equilibrium wage depends upon the realization of the technology and the preference shocks. Similarly, before the state of nature is realized, households – who own the capital stock – and firms trade state-contingent rental contracts on capital which specify the quantity traded and the rental rate as functions of the information set \( \Gamma_t \equiv (\omega_t, z_t) \). With this specification, the ex-post rental rate depends upon the realized rate of capital utilization through its dependence on the shocks to technology and preferences.

A.1 Households

For simplicity, I state the household’s problem the same way as in Section 3.\(^{23}\) The typical household solves the following problem,

\[
V(\Phi_t) = \max_{N_t} \max_{\Theta_t} E \left\{ \max_{\Omega_t} E \left\{ u(C_t, l_t, e_t, Q_t; z_t) + \beta V(\Phi_{t+1}) \mid \Gamma_t \mid \Gamma_{t-1} \right\} \right\}
\]

\(^{22}\)This specification was also employed by Burnside et al. (1993) and Burnside and Eichenbaum (1996) in developing the factor-hoarding models.

\(^{23}\)To be strictly consistent with Hansen (1985) and Rogerson (1988), one should write the household’s work-leisure decision in terms of the probability of working. However, one can see below that factor prices and quantities are, at the end, determined the same way as in those models. this occurs because I reduce the infinite-agent-with-unit-mass problem to a single-agent problem with linear utility in leisure as the one described in Rogerson (1988). This simplification is made for the sake of describing a simple labor market with standard supply and demand schedules.
subject to (1) – the law of motion of capital -, (14) – the specific utility function -, the
shocks processes, and her budget constraint given by,

\[ C_t + I_t[1 + h(I_t)] + Q_{t+1} - Q_t \leq w_t e_t N_t + r_t s_t K_t, \]  

(A.2)

where, \( w_t \) and \( r_t \) are the wage and rental rate on labor and capital services, both given to
the household.

The first order conditions are

\[ E\left[u_1(C_t, l_t, e_t, Q_t)w_t - u_2(C_t, l_t, e_t, Q_t) | \Gamma_{t-1}\right] = 0 \]  

(A.3)

\[ u_1(C_t, l_t, e_t, Q_t)w_t - u_2(C_t, l_t, e_t, Q_t) = -\frac{u_3(C_t, l_t, e_t, Q_t)}{N_t} \]  

(A.4)

\[ r_t = \delta'(s_t)p_{k,t} \]  

(A.5)

\[ u_1(C_t, l_t, e_t, Q_t)p_{k,t} = \beta E \left\{ \frac{u_1(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1})}{[s_{t+1} r_{t+1} + (1 - \delta_{t+1})p_{k,t+1}] | \Gamma_t} \right\} \]  

(A.6)

\[ u_1(C_t, l_t, e_t, Q_t) = \beta E \left[ \frac{u_1(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1}) + u_4(C_{t+1}, l_{t+1}, e_{t+1}, Q_{t+1})}{| \Gamma_t} \right] \]  

(A.7)

The first equation is the necessary condition on the labor supply as the extensive margin, where the wage rate is written inside the expectation operator to indicate that contracts are written conditional on the realization of the shocks. The second expression refers to the decision on the intensity of use of labor, where the wage rate is now denoted \textit{ex-post}. Notice that what is going to ultimately define the amount of labor services supplied by the household will be the decision related to the intensity of use of labor, at the \textit{ex-post} wage, \( w_t \). The third condition is related to the rate of capital utilization, which, as in the case of the labor supply, is going to define the supply of capital services by the household, given the \textit{ex-post} rental rate, \( r_t \). Conditions four and five are related to the decisions on investment both in capital and inventories.

A.2 Firms

The firm’s problem can be stated as follows. At every date \( t \), the firm maximizes expected profits by

\[ \max_{(L_t, S_t)} E_t \left[ F(L_t, S_t; \omega_t) - w_t L_t + r_t S_t | \Gamma_{t-1} \right] \]

subject to the Cobb-Douglas production technology given by \( F(L_t, S_t; \omega_t) = \omega_t L_t^{1-\alpha} S_t^\alpha \) – a modified version of (12) indicating that the firm chooses to rent expected labor and capital.
services according to households’ supply, at the expected factor prices \( w_t \) and \( r_t \).\(^{24}\)

The first order conditions are

\[
E_t \left[ F_1(L^d_t, S^d_t) - w_t \mid \Gamma_{t-1} \right] = 0 \quad (A.8)
\]

\[
E_t \left[ F_2(L^d_t, S^d_t) - r_t \mid \Gamma_{t-1} \right] = 0 \quad (A.9)
\]

That is, the expected marginal products of labor services and capital services must equal the \textit{ex-ante} wage and rental rates.

\textit{Ex-post}, the firm’s best response is to choose labor and capital services according to

\[
F_1(L^d_t, S^d_t) - w_t = 0 \quad (A.10)
\]

\[
F_2(L^d_t, S^d_t) - r_t = 0 \quad (A.11)
\]

### A.3 Equilibrium

Define the competitive equilibrium as a sequence \( \{C_t, Q_t, K_t, N_t, s_t, e_t, w_t, r_t\}_{t=0}^{\infty} \) so that households and firms solve their optimization problems, and the markets for factors of production and goods clear.

Condition (A.3) together with (A.8), and condition (A.6) together with (A.9) determine the trade of state-contingent contracts on labor and capital services at the expected factor prices \( E_t\{w_t \mid \Gamma_{t-1}\} \) and \( E_t\{r_t \mid \Gamma_{t-1}\} \). Once the states of technology and preferences are realized, equilibria in the markets for labor and capital services need that \( L^d_t = e_t N_t \) and \( S^d_t = s_t K_t \). This implies that the equilibrium wage and rental rate must satisfy conditions (A.4)-(A.5) and (A.10)-(A.11). This will be held indeed for contracts are state-contingent, which means that the firm commits to pay wage and rental rate according to the realized marginal products of labor and capital services. Otherwise, it would not be an equilibrium to the firm.

Naturally, the two equations describing the equilibrium quantities of labor and capital services will be the same as those found in the representative-agent problem of Section 3, namely, conditions (8) and (9). Finally, equilibrium in the output market is obtained from the market clearing condition:

\[
C_t + I_t[1 + h(I_t)] + Q_{t+1} - Q_t = y_t. \quad (A.12)
\]

To sum up, the market-specification could be described through the following steps:

\(^{24}\)Notice that the expected output price was normalized to 1, so that the relative price of capital to output, \( p_k \), is as specified in Section 3. Remember that in the steady state, \( p_k > 1 \), for the steady state rate of transformation from output to capital is positive.
1. At the beginning of each period, households and firms trade contingent contracts on labor and capital services, according to expected prices and quantities.

2. These contracts specify, first, the amounts of labor and capital, $N_t$ and $K_t$, supplied with certainty by the household.

3. Given these stocks of labor and capital, the decision on the services is a decision on the rates of use of these productive factors, and I assume these are decided by the household.

4. Contingent contracts imply a commitment on behalf of the firm to pay wage and rental rate at the end of the period, according to the services received.

5. Equilibrium prices and quantities of these services ultimately depend on the firm’s maximizing behavior. For it is the firm who specify, ex-ante, a state-contingent plan where, for any bundle of factor services $(L_t, S_t)$, it is willing to pay $w_t(\omega_t, z_t)L_t + r_t(\omega_t, z_t)S_t$.

6. After the vector of states of technology and preferences $(\omega_t, z_t)$ is realized, there is no trade between firm and household. The household optimally chooses the rates of labor and capital utilization according to the plan traded with the firm. Finally, equilibrium quantities of factor services and prices result.

B Modeling details

B.1 System of necessary conditions, particular forms:

\[ E \left\{ e_t \left[ z_t \left( \theta C_t^{1-\gamma} + (1 - \theta)Q_t^{1-\gamma} \right)^{-1} \theta C_t^{-\gamma}(1 - \alpha) \left( \frac{Y_t}{e_t N_t} \right) - \eta \right] \mid \Gamma_{t-1} \right\} = 0 \]  

\[ N_t \left[ z_t \left( \theta C_t^{1-\gamma} + (1 - \theta)Q_t^{1-\gamma} \right)^{-1} \theta C_t^{-\gamma}(1 - \alpha) \left( \frac{Y_t}{e_t N_t} \right) - \eta \right] = \phi(e_t - 1) \]  

\[ \alpha \left( \frac{Y_t}{s_t K_t} \right) = v s_t^{(v-1)} (1 + 2b I_t) \]

\[ z_t \left( \theta C_t^{1-\gamma} + (1 - \theta)Q_t^{1-\gamma} \right)^{-1} \theta C_t^{-\gamma}(1 + 2b I_t) = \beta E \left\{ z_{t+1} \left( \theta C_{t+1}^{1-\gamma} + (1 - \theta)Q_{t+1}^{1-\gamma} \right)^{-1} \theta C_{t+1}^{-\gamma} \times \right\} \]

\[ s_{t+1} \left[ \frac{Y_{t+1}}{s_{t+1} K_{t+1}} \right] + \left[ 1 - (\delta_0 + s_{t+1}^\eta)(1 + 2b I_{t+1}) \right] \mid \Gamma_t \]

45
\[ z_t \left( \theta C_t^{1-\gamma} + (1 - \theta)Q_t^{1-\gamma} \right)^{-1} \theta C_t^{-\gamma} = \beta E \left\{ z_{t+1} \left[ \theta C_{t+1}^{-\gamma} + (1 - \theta)Q_{t+1}^{-\gamma} \right] | \Gamma_t \right\} \]  
(B.5)

\[ C_t + I_t(1 + bI_t) + Q_{t+1} - Q_t = Y_t \]  
(B.6)

\[ K_{t+1} = K_t[1 - (\delta_0 + s_t^v)] + I_t \]  
(B.7)

### B.2 Steady-State Expressions:

\[ s = \left[ \frac{1 - \beta(1 - \delta_0)}{\beta(v - 1)} \right]^{1/v} \]  
(B.8)

\[ \delta = \delta_0 + s^v \]  
(B.9)

\[ AB[D(1 + 2b\delta K)]^{v\delta - 1} - K(sD - \delta) - K^2b\delta(2sD - \delta) = 0 \]  
(B.10)

\[ N = sK \left[ D(1 + 2b\delta K) \right]^{\frac{1}{1-\alpha}} \]  
(B.11)

\[ Q = B \left( \frac{sK}{N} \right)^\alpha \]  
(B.12)

\[ C = AQ \]  
(B.13)

\[ I = \delta K \]  
(B.14)

where,

\[ A \equiv \left[ \frac{(1 - \beta)(1 - \delta_0)}{(1 - \theta)\beta} \right]^{1/\gamma} \]  
(B.15)

\[ B \equiv \frac{(1 - \alpha)(1 - \theta)}{[\theta A^{1-\gamma} + (1 - \theta)]\eta(1 - \beta)} \]  
(B.16)

\[ D \equiv \frac{vs^v}{\alpha} \]  
(B.17)

### B.3 Baseline Parameters:

<table>
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<th>$\beta$</th>
<th>$\delta_0$</th>
<th>$v$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\theta$</th>
<th>$\eta$</th>
<th>$\phi$</th>
<th>$b$</th>
<th>$\sigma_z$</th>
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<td>.01</td>
<td>.007</td>
<td>.95</td>
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C Figures and Tables

Figure 1: Timeline of Decisions and Information

Figure 2: The effect of the rate of labor utilization

Figure 3: High-Persistence Shocks. Inventories, Labor Utilization, Capital Utilization (% deviations from steady state)
Figure 4: Low-Persistence Shocks. Inventories, Labor Utilization, Capital Utilization (% deviations from steady state)

Figure 5: High-Persistence Shocks. Labor and Capital Services (% deviations from steady state)

Figure 6: Low-Persistence Shocks. Labor and Capital Services (% deviations from steady state)
Figure 7: High-Persistence Preference Shock. Effects of Inventories and Capacity Utilization (% deviations from steady state)

Figure 8: Low-Persistence Preference Shock. Effects of Inventories and Capacity Utilization (% deviations from steady state)

Figure 9: High-Persistence Technology Shock. Effects of Inventories and Capacity Utilization (% deviations from steady state)

Figure 10: Low-Persistence Technology Shock. Effects of Inventories and Capacity Utilization (% deviations from steady state)
Figure 11: Stochastic Steady State Values

Figure 12: CEE Vs. Benchmark (% deviations from steady state)
Table 1: U.S. Inventory and Capacity Utilization Facts

<table>
<thead>
<tr>
<th></th>
<th>Inventories</th>
<th>Inventory/Sales</th>
<th>Capital Services</th>
<th>Labor Services</th>
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<tr>
<td>Volatility relative to GNP</td>
<td>0.93</td>
<td>0.84</td>
<td>2.06</td>
<td>0.83</td>
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<tr>
<td>Correlation with GNP</td>
<td>0.57</td>
<td>(0.39)</td>
<td>0.90</td>
<td>0.88</td>
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<tr>
<td>Correlation with Inventories</td>
<td>-</td>
<td>0.51</td>
<td>0.56</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Data are quarterly U.S. 1967:1-2008:4 seasonally adjusted, filtered (HP, 1600) series. The beginning date is the first for which the Federal Reserve Board's capacity utilization series is available. Inventories are real private inventories and Sales are final sales of domestic product, both in 2000 chained dollars. Capital Services was constructed from the industry index of capacity utilization and the real stock of capital series by the Bureau of Economic Analysis. Labor Services was constructed from the average weekly hours and employment series (Bureau of Labor Statistics).

Table 2: Percent Standard Deviations Relative to Output: Benchmark Specification

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<tr>
<th></th>
<th>Baseline</th>
<th>No Inventories</th>
<th>No Variable Capacity Util.</th>
<th>No Variable Capital Util.</th>
<th>No Variable Labor Util.</th>
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<td>Capital</td>
<td>0.16</td>
<td>0.18</td>
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<td>Capital Services</td>
<td>0.40</td>
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<td>-</td>
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<tr>
<td>Capital Utilization</td>
<td>0.41</td>
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<td>0.42</td>
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<td>Consumption</td>
<td>0.64</td>
<td>0.74</td>
<td>0.67</td>
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<td>Depreciation</td>
<td>0.57</td>
<td>0.54</td>
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<td>-</td>
<td>0.58</td>
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<td>Invent.-Sales</td>
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<td>0.90</td>
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<td>Inventories</td>
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<td>Investment</td>
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<td>Labor</td>
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<td>0.78</td>
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<tr>
<td>Labor Services</td>
<td>0.80</td>
<td>0.74</td>
<td>-</td>
<td>0.87</td>
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</tr>
<tr>
<td>Labor Utilization</td>
<td>0.24</td>
<td>0.30</td>
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<td>0.31</td>
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Table 3: Correlation Matrix: Benchmark Specification

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<td>0.17</td>
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<td>0.16</td>
<td>(0.10)</td>
<td>0.16</td>
<td>(0.10)</td>
<td>(0.05)</td>
<td>(0.13)</td>
</tr>
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<td>0.91</td>
<td>0.93</td>
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<td>0.87</td>
<td>1.00</td>
<td>0.51</td>
<td>0.69</td>
<td>(0.75)</td>
<td>0.82</td>
<td>0.84</td>
<td>0.93</td>
<td>0.16</td>
<td>0.97</td>
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<td>0.56</td>
<td>0.34</td>
<td>(0.71)</td>
<td>0.68</td>
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<td>0.88</td>
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<td>(0.75)</td>
<td>0.82</td>
<td>0.84</td>
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<td>(0.35)</td>
<td>(0.63)</td>
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<td></td>
</tr>
<tr>
<td>Invent.-Sales</td>
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<td>(0.67)</td>
<td>(0.86)</td>
<td>(0.56)</td>
<td>(0.75)</td>
<td>(0.72)</td>
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</table>

Table 4: Correlation among Inventories, Capital Utilization, and Labor Utilization: Benchmark Specification

<table>
<thead>
<tr>
<th>Preference shock</th>
<th>Non-persistent Shock</th>
<th>Technology shock</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inventories</td>
<td>1.00</td>
<td>-0.84</td>
</tr>
<tr>
<td>Rate Cap. Utiliz.</td>
<td>1.00</td>
<td>0.47</td>
</tr>
<tr>
<td>Rate Labor Utiliz.</td>
<td>1.00</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Persistent Shock</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inventories</td>
</tr>
<tr>
<td>Rate Cap. Utiliz.</td>
</tr>
<tr>
<td>Rate Labor Utiliz.</td>
</tr>
</tbody>
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Table 5: Correlation among Inventories, Capital Utilization, and Labor Utilization: Isolating Effects

<table>
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<th></th>
<th>Preference shock</th>
<th>Non-persistent Shock</th>
<th>Technology shock</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inventories</td>
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<td>-0.60</td>
<td>-0.82</td>
</tr>
<tr>
<td>Rate Cap. Utiliz.</td>
<td>1.00</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>Rate Labor Utiliz.</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>

Note: Each correlation between two variables implies that the third one is absent from the specification.

Table 6: Rate of Capital Utilization: Coefficients of Autocorrelation

<table>
<thead>
<tr>
<th>Order</th>
<th>Benchmark</th>
<th>CEE</th>
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<tr>
<td>1</td>
<td>0.787</td>
<td>0.800</td>
</tr>
<tr>
<td>2</td>
<td>0.460</td>
<td>0.494</td>
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<tr>
<td>3</td>
<td>0.230</td>
<td>0.262</td>
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<tr>
<td>4</td>
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<td>0.085</td>
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<tr>
<td>5</td>
<td>-0.057</td>
<td>-0.046</td>
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</table>
Table 7: Percent Standard Deviations Relative to Output: CEE Specification

<table>
<thead>
<tr>
<th></th>
<th>Baseline</th>
<th>No Inventories</th>
<th>No Variable Capacity Util.</th>
<th>No Variable Capital Util.</th>
<th>No Variable Labor Util.</th>
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<tbody>
<tr>
<td>Capital</td>
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<td>0.20</td>
<td>0.21</td>
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<tr>
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<td>0.51</td>
<td>-</td>
<td>-</td>
<td>0.50</td>
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<tr>
<td>Capital Utilization</td>
<td>0.51</td>
<td>0.51</td>
<td>-</td>
<td>-</td>
<td>0.51</td>
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<tr>
<td>Consumption</td>
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<td>0.65</td>
<td>0.69</td>
<td>0.70</td>
<td>0.54</td>
</tr>
<tr>
<td>Invent.-Sales</td>
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<td>-</td>
<td>0.90</td>
<td>0.80</td>
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<td>-</td>
<td>0.56</td>
<td>0.46</td>
<td>0.54</td>
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<td>1.59</td>
<td>1.72</td>
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Table 8: Correlation Matrix: CEE Specification

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<td>(0.43)</td>
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